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# SOME SINGULAR MODULI (I)

*By* G. N. WATSON (*Birmingham*)

[Received 5 September 1931]

ALTHOUGH lists of values of singular moduli occupy only a comparatively small portion of Ramanujan's note-books and published works, he must have spent a fairly lengthy period in computing them. An estimate of the extent of his labours may be formed from the verifications in this paper of some of his results.

The expressions calculated by Ramanujan are defined (in the ordinary notation of elliptic functions) by the equations

$$q = e^{-\pi\sqrt{n}},$$

$$G_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}(1+q)(1+q^3)(1+q^5)\ldots = (2kk')^{-\frac{1}{6}},$$

$$g_n = 2^{-\frac{1}{4}}q^{-\frac{1}{24}}(1-q)(1-q^3)(1-q^5)\ldots = \left(\frac{k'^2}{2k}\right)^{\frac{1}{8}},$$

for various integral values of $n$. For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine $G_n$ for odd values of $n$, and $g_n$ for even values of $n$.

It is to be remarked that, when $n$ is an unrestricted positive number,

$$G_n = G_{1/n}, \qquad g_n = 1/g_{4/n};$$

and that $G_n$ is an increasing function of $n$ when $n > 1$, while $g_n$ always increases with $n$.

The following table summarizes an analysis of the number of values of $n$ for which values of $G_n$ or $g_n$ are extant either in certain of Ramanujan's writings or in the list at the end of Weber's *Algebra*:*

| | |
|---|---|
| $N+Q = 21$ | $Q = 25$ |
| $N+W = 56$ | $W = 49$ |

Here $Q$ indicates the number which appear only in Ramanujan's paper in the *Quarterly Journal*,† and $W$ the number which appear

* Weber omitted from his list a few singular moduli whose values were known at the date of its publication (1908); several other singular moduli have been computed since.

† *Quart. J. of Math.* 45 (1914), 350–72.

only in Weber's *Algebra*; $N+Q$ indicates the number common to the second edition of the note-book* and the paper in the *Quarterly Journal*, and similarly for $N+W$. Ramanujan became acquainted with Weber's *Algebra* before the publication of his paper and deliberately omitted from it all values given by Weber, so the combination $W+Q$ does not occur.

It will be seen that Ramanujan constructed 102 singular moduli in all, of which 46 were not given by Weber. It is intended to publish the calculations involved in the construction of the set $N+Q$ as part of the commentary on the note-books by Dr. B. M. Wilson and myself; in this and in a subsequent paper I propose to investigate the set $Q$, with the exception of $G_{1353}$ which has been discussed elsewhere.† It may be mentioned that many, if not all, of the members of the sets $Q$, $N+Q$, and $N+W$ are to be found in the first edition of the note-book, but the arrangement of them there is quite unsystematic.

Of the twenty-four remaining members of $Q$, I believe that fourteen were obtained by Ramanujan by means of the empirical process which I described in the discussion of $G_{1353}$. In the present paper I investigate these fourteen, namely, $G_{333}$, $G_{465}$, $G_{765}$, $G_{777}$, $G_{1645}$, $g_{66}$, $g_{114}$, $g_{126}$, $g_{138}$, $g_{154}$, $g_{238}$, $g_{310}$, $g_{522}$, $g_{630}$. To them I have added $G_{897}$ and $G_{1677}$, making sixteen in all. These last two moduli are, I believe, new results; there is a brief statement about them in the first edition of the note-book‡ which confirms my belief that the empirical process was Ramanujan's normal method of attack, though it is not impossible that he used other methods partially or wholly in the determination of $g_{66}$, $g_{154}$, and (less probably) $G_{333}$, $G_{765}$, $g_{126}$, $g_{522}$, $g_{630}$.

In the subsequent paper I shall discuss the remainder of the set $Q$, namely, $G_{121}$, $G_{169}$, $G_{225}$, $G_{325}$, $G_{363}$, $G_{1225}$, $g_{62}$, $g_{94}$, $g_{98}$, $g_{158}$, and to them I shall add $g_{254}$; the values of these can be obtained rigorously without an undue amount of labour by means of knowledge which was at Ramanujan's disposal.

In the present paper details of computations have been suppressed as far as possible; I state merely the results of computing expressions of the forms $\log_{10} G_n$ and $\log_{10} g_n$ by means of $q$-products and the

* In this analysis I am completely ignoring the first edition of the note-book.

† *Journal London Math. Soc.* 6 (1931), 126–32.

‡ There seem to be no other moduli in the first edition which are not to be found elsewhere in the writings of Ramanujan or other mathematicians.

values of combinations of such expressions which seem likely to be capable of identification with simple surds; the relevant surds are usually taken from Barlow's *Tables* and are therefore given to nine significant figures, other computations being given to ten significant figures of which the last is doubtful. The value adopted for $\pi \log_{10} e$ in computing logarithms of $q$-products is

$$1{\cdot}36437\ 63538\ 41841.$$

The number of the quadratic equations which have to be solved in the course of the paper is so great that it is expedient to state here once for all that the product of the roots of each quadratic is 1 and that the roughest of numerical calculations shows that in each case the greater root is the root which is required.

At the beginning of the calculation of each $G_n$ and $g_n$ I have noted the generic character and class-number of the negative determinant $-n$, as given by Gauss, *Werke*, 2, 450–74; these indicate the nature of the irrationalities to be anticipated.

Notwithstanding the empirical nature of the paper, numerous (but not always complete) checks on the work can be supplied by using modular equations of quite small orders; at least one such check has been applied to each of the singular moduli here discussed. For example, in the investigations of $G_{465}$ and $G_{777}$, it can be deduced from modular equations of order 3 that

$$8\left\{(xyzw)^3+\frac{1}{(xyzw)^3}\right\}$$
$$=8\left\{\left(\frac{xy}{zw}\right)^3+\left(\frac{zw}{xy}\right)^3\right\}+\left\{\left(\frac{xz}{yw}\right)^6+\left(\frac{yw}{xz}\right)^6\right\}+\left\{\left(\frac{xw}{yz}\right)^6+\left(\frac{yz}{xw}\right)^6\right\};$$

in dealing with $G_{465}$ it happened to be simplest to guess the four surds and verify that they satisfied this equation, while with $G_{777}$ it was less troublesome to carry out the equivalent process of guessing three of the surds and deducing the value of the remaining surd from the equation just given.

(i) $\mathbf{n = 333 = 3^2 \times 37}$. (G.IV, class-number 2.)

Let $G_{333} = x$, $G_{333/9} = y$, $G_{333/81} = z$.

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}96213\ 96714, \qquad \log_{10} z = 0{\cdot}04075\ 19094,$$
$$\log_{10} y = 0{\cdot}27054\ 15600.$$

We thus have

$$\frac{x}{z}+\frac{z}{x} = 8{\cdot}46410\,1615, \qquad 5+2\sqrt{3} = 8{\cdot}46410\,162,$$

$$y^4 = 12{\cdot}08276\,253, \qquad \sqrt{37}+6 = 12{\cdot}08276\,253,$$

$$\frac{xz}{y^2}+\frac{y^2}{xz} = 3{\cdot}24136\,2032.$$

The last expression does not suggest a simple surd; but

$$\left(\frac{xz}{y^2}\right)^3+\left(\frac{y^2}{xz}\right)^3 = 24{\cdot}33105\,010, \qquad \sqrt{592} = 24{\cdot}33105\,01.$$

Assume the truth of the equalities suggested by these results;* by solving quadratic equations, we then get

$$\frac{x}{z} = \left\{\sqrt{\left(\frac{7+2\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{3}}{4}\right)}\right\}^2$$

$$\left(\frac{xz}{y^2}\right)^3 = 2\sqrt{37}+7\sqrt{3}.$$

Hence

$$G_{333} = (\sqrt{37}+6)^{\frac{1}{4}}(2\sqrt{37}+7\sqrt{3})^{\frac{1}{4}}\left\{\sqrt{\left(\frac{7+2\sqrt{3}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{3}}{4}\right)}\right\}.$$

It is within the bounds of possibility that Ramanujan obtained this result from the modular equation of order 3

$$\frac{G_{333}^6}{G_{37}^6}+\frac{G_{37}^6}{G_{333}^6} = 2\sqrt{2}.\left(G_{333}^3G_{37}^3-\frac{1}{G_{333}^3G_{37}^3}\right)$$

after obtaining $G_{37}$ empirically; the task of solving such a quartic is more tedious (in the case of negative determinants with the classification IV. 2) than the empirical method which is used here. A similar remark applies to $G_{765}$, $g_{126}$, $g_{522}$, and $g_{630}$.

(ii) $\mathbf{n = 465 = 3\times5\times31}$. (G. VIII, class-number 2.)

Let

$$G_{465} = x, \qquad G_{465/9} = y, \qquad G_{465/25} = z, \qquad G_{465/225} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10}x = 1{\cdot}15062\,66196, \qquad \log_{10}y = 0{\cdot}33337\,05406,$$

$$\log_{10}z = 0{\cdot}16991\,98917, \qquad \log_{10}w = 0{\cdot}01118\,93849.$$

* The value assumed for $G_{37}$ agrees with the value given in Weber's *Algebra*. Ramanujan probably obtained it empirically because his note-book contains no modular equation of order 37.

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 20{\cdot}13552\,873, \qquad 9+2\sqrt{31} = 20{\cdot}13552\,87,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 9{\cdot}56776\,4365, \qquad 4+\sqrt{31} = 9{\cdot}56776\,436,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 4{\cdot}77491\,0254,$$

$$xyzw+\frac{1}{xyzw} = 46{\cdot}27105\,745, \qquad 24+4\sqrt{31} = 46{\cdot}2710575.$$

The third of the four expressions does not suggest any simple surd; but

$$\left(\frac{xw}{yz}\right)^3+\left(\frac{yz}{xw}\right)^3 = 94{\cdot}54211\,493, \qquad 50+8\sqrt{31} = 94{\cdot}54211\,49.$$

Assume the truth of the four equalities suggested by these results and write

$$A = \frac{9+2\sqrt{31}+\sqrt{(201+36\sqrt{31})}}{2} = (2+\sqrt{3})\left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right),$$

$$B = \frac{4+\sqrt{31}+\sqrt{(43+8\sqrt{31})}}{2} = \left\{\sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}+\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)}\right\}^2,$$

$$C = 25+4\sqrt{31}+\sqrt{(1120+200\sqrt{31})} = (5\sqrt{5}+2\sqrt{31})\left(\frac{\sqrt{5}+1}{2}\right)^3,$$

$$D = 12+2\sqrt{31}+\sqrt{(267+48\sqrt{31})} = \left\{\sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}+\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)}\right\}^2.$$

Then, by solving quadratic equations, we have

$$\frac{xy}{zw} = A, \qquad \frac{xz}{yw} = B, \qquad \left(\frac{xw}{yz}\right)^3 = C, \qquad xyzw = D,$$

whence

$$x^4 = AB^{\frac{1}{2}}C^{\frac{1}{2}}D,$$

that is to say,

$$G_{465}^2 = \sqrt{\left\{(2+\sqrt{3})\left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)\left(\frac{\sqrt{5}+1}{2}\right)\right\}}.(5\sqrt{5}+2\sqrt{31})^{\frac{1}{2}}\times$$
$$\times\left\{\sqrt{\left(\frac{6+\sqrt{31}}{4}\right)}+\sqrt{\left(\frac{2+\sqrt{31}}{4}\right)}\right\}\times$$
$$\times\left\{\sqrt{\left(\frac{13+2\sqrt{31}}{2}\right)}+\sqrt{\left(\frac{11+2\sqrt{31}}{2}\right)}\right\}.$$

(iii) $\mathbf{n = 765 = 3^2 \times 5 \times 17}$. (G. VIII, class-number 2.)

The repeated factor of 765 makes the choice of the unknowns slightly less obvious than usual. Let

$$G_{765} = x, \qquad G_{765/25} = y, \qquad G_{765/81} = z, \qquad G_{765/2025} = G_{765/289} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 1{\cdot}49710\,85574, \qquad \log_{10} y = 0{\cdot}23921\,57246,$$
$$\log_{10} z = 0{\cdot}09947\,76888, \qquad \log_{10} w = 0{\cdot}01984\,48804.$$

We thus have

$$\frac{xy}{zw} + \frac{zw}{xy} = 41{\cdot}42428\,527, \qquad 20 + 3\sqrt{51} = 41{\cdot}42428\,53,$$

$$\frac{xz}{yw} + \frac{yw}{xz} = 21{\cdot}79929\,462,$$

$$\frac{xw}{yz} + \frac{yz}{xw} = 15{\cdot}14142\,843, \qquad 8 + \sqrt{51} = 15{\cdot}14142\,84,$$

$$xyzw + \frac{1}{xyzw} = 71{\cdot}73502\,796.$$

The second and fourth of the four expressions do not suggest any simple surds; but

$$\left(\frac{xz}{yw}\right)^3 + \left(\frac{yw}{xz}\right)^3 = 10293{\cdot}82848, \qquad 5152 + 720\sqrt{51} = 10293{\cdot}82847,$$

$$xyzw - \frac{1}{xyzw} = 71{\cdot}70714\,216, \qquad 36 + 5\sqrt{51} = 71{\cdot}70714\,21.$$

Assume the truth of the four equalities suggested by these results; by solving quadratic equations, we then get

$$\frac{xy}{zw} = \left\{\sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)} + \sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)}\right\}^2,$$

$$\left(\frac{xz}{yw}\right)^3 = (2576+360\sqrt{51}) + (161\sqrt{255}+1152\sqrt{5})$$
$$= (16+\sqrt{255})\left(\frac{3+\sqrt{5}}{2}\right)^6,$$

$$\frac{xw}{yz} = \left\{\sqrt{\left(\frac{10+\sqrt{51}}{4}\right)} + \sqrt{\left(\frac{6+\sqrt{51}}{4}\right)}\right\}^2,$$

$$xyzw = \tfrac{1}{2}(36+5\sqrt{51}) + \tfrac{1}{2}(9\sqrt{15}+4\sqrt{85})$$
$$= (4+\sqrt{5})\left(\frac{\sqrt{85}+9}{2}\right).$$

Hence $$G_{765}^2 = \left(\frac{\sqrt{5}+1}{2}\right)^2\left(\frac{\sqrt{17}+\sqrt{15}}{\sqrt{2}}\right)^{\frac{1}{2}}\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)\sqrt{\left(\frac{\sqrt{85}+9}{2}\right)}\times$$
$$\times\left\{\sqrt{\left(\frac{22+3\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{18+3\sqrt{51}}{4}\right)}\right\}\times$$
$$\times\left\{\sqrt{\left(\frac{10+\sqrt{51}}{4}\right)}+\sqrt{\left(\frac{6+\sqrt{51}}{4}\right)}\right\}.$$

(iv) $\mathbf{n = 777 = 3\times 7\times 37}$. (G. VIII, class-number 2.)

Let

$$G_{777} = x, \qquad G_{777/9} = y, \qquad G_{777/49} = z, \qquad G_{777/441} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 1{\cdot}50939\,28540, \qquad \log_{10} y = 0{\cdot}45295\,92854,$$
$$\log_{10} z = 0{\cdot}15112\,27250, \qquad \log_{10} w = 0{\cdot}00686\,28355.$$

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 63{\cdot}74901\,574, \qquad 32+12\sqrt{7} = 63{\cdot}74901\,57,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 15{\cdot}93725\,393, \qquad 8+3\sqrt{7} = 15{\cdot}93725\,393,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 8{\cdot}29150\,2626, \qquad 3+2\sqrt{7} = 8{\cdot}29150\,262,$$

$$xyzw+\frac{1}{xyzw} = 131{\cdot}93579\,90.$$

The last of the four expressions does not suggest any simple surd; but

$$(xyzw)^3+\frac{1}{(xyzw)^3} = 22\,96217{\cdot}908,$$

$$11\,48110+4\,33944\sqrt{7} = 22\,96217{\cdot}907.$$

Assume the truth of the equalities suggested by the first three results; the truth of the fourth suggested equality is then confirmed by the modular equation of order 3; by solving quadratic equations, we then get

$$\frac{xy}{zw} = \left\{\sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)}\right\}^2,$$

$$\frac{xz}{yw} = \left\{\sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)}\right\}^2,$$

$$\frac{xw}{yz} = \tfrac{1}{2}(3+2\sqrt{7})+\tfrac{1}{2}(\sqrt{21}+2\sqrt{3})$$

$$= (2+\sqrt{3})\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right),$$

$$(xyzw)^3 = (5\,74055+2\,16972\sqrt{7})+(94374\sqrt{37}+35670\sqrt{259})$$

$$= (107\sqrt{37}+246\sqrt{7})(\sqrt{37}+6)^3.$$

Hence

$$G_{777}^2 = \sqrt{\left\{(2+\sqrt{3})\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)(\sqrt{37}+6)\right\}}.(107\sqrt{37}+246\sqrt{7})^{\frac{1}{4}}\times$$

$$\times\left\{\sqrt{\left(\frac{17+6\sqrt{7}}{2}\right)}+\sqrt{\left(\frac{15+6\sqrt{7}}{2}\right)}\right\}\times$$

$$\times\left\{\sqrt{\left(\frac{10+3\sqrt{7}}{4}\right)}+\sqrt{\left(\frac{6+3\sqrt{7}}{4}\right)}\right\}.$$

(v) $\mathbf{n = 897 = 3\times 13\times 23}$. (G. VIII, class-number 2.)

Let

$$G_{897} = x, \qquad G_{897/9} = y, \qquad G_{897/169} = z, \qquad G_{897/529} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 1{\cdot}62736\,81200, \qquad \log_{10} y = 0{\cdot}49228\,43741,$$

$$\log_{10} z = 0{\cdot}05602\,58210, \qquad \log_{10} w = 0{\cdot}00597\,49247.$$

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 114{\cdot}20498\,21, \qquad 58+9\sqrt{39} = 114{\cdot}20498\,20,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 15{\cdot}38091\,492,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 12{\cdot}24499\,800, \qquad 6+\sqrt{39} = 12{\cdot}24499\,800,$$

$$xyzw+\frac{1}{xyzw} = 151{\cdot}93997\,59, \qquad 77+12\sqrt{39} = 151{\cdot}93997\,60.$$

The second of the four expressions does not suggest any simple surd; but

$$\left(\frac{xz}{yw}\right)^6+\left(\frac{yw}{xz}\right)^6 = 129\,06481{\cdot}21,$$

$$64\,53250+10\,33344\sqrt{39} = 129\,06481{\cdot}212.$$

Assume the truth of the equalities* suggested by the first three results; the truth of the fourth suggested equality is then confirmed by the modular equation of order 3; by solving quadratic equations, we then get

$$\frac{xy}{zw} = \left\{\sqrt{\left(\frac{60+9\sqrt{39}}{4}\right)}+\sqrt{\left(\frac{56+9\sqrt{39}}{4}\right)}\right\}^2,$$

$$\left(\frac{xz}{yw}\right)^6 = (32\,26625+5\,16672\sqrt{39})+(3\,88440\sqrt{69}+1\,86600\sqrt{299})$$

$$= (7775+936\sqrt{69})(415+24\sqrt{299})$$

$$= \left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^6(4\sqrt{13}+3\sqrt{23})^2,$$

$$\frac{xw}{yz} = \left\{\sqrt{\left(\frac{8+\sqrt{39}}{4}\right)}+\sqrt{\left(\frac{4+\sqrt{39}}{4}\right)}\right\}^2,$$

$$xyzw = \tfrac{1}{2}(77+12\sqrt{39})+\tfrac{1}{2}(44\sqrt{3}+21\sqrt{13})$$

$$= (2+\sqrt{3})^2\left(\frac{\sqrt{13}+3}{2}\right)^2.$$

Hence we have the new result

$$G_{897}^2 = (2+\sqrt{3})\left(\frac{\sqrt{13}+3}{2}\right)(4\sqrt{13}+3\sqrt{23})^{\frac{1}{2}}\sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}\times$$
$$\times\left\{\sqrt{\left(\frac{60+9\sqrt{39}}{4}\right)}+\sqrt{\left(\frac{56+9\sqrt{39}}{4}\right)}\right\}\times$$
$$\times\left\{\sqrt{\left(\frac{8+\sqrt{39}}{4}\right)}+\sqrt{\left(\frac{4+\sqrt{39}}{4}\right)}\right\}.$$

(vi) $\mathbf{n = 1645} = 5\times7\times47$. (G. VIII, class-number 2.)

Let

$$G_{1645} = x, \qquad G_{1645/25} = y, \qquad G_{1645/49} = z, \qquad G_{1645/1225} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10}x = 2{\cdot}23045\,89266, \qquad \log_{10}y = 0{\cdot}38588\,57862,$$
$$\log_{10}z = 0{\cdot}25413\,05672, \qquad \log_{10}w = 0{\cdot}00187\,62238.$$

* In his note-book all that Ramanujan wrote under the heading $\sqrt{897}$ was merely

$$Q+\frac{1}{Q} = 58+9\sqrt{39}, \qquad S+\frac{1}{S} = 6+\sqrt{39}.$$

This statement and the corresponding statement for $\sqrt{1677}$ are the only clues to his methods which exist. They obviously corroborate my opinion that his methods were substantially the methods of this paper.

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 229{\cdot}2694474,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 124{\cdot}9842500, \qquad \frac{123+7\sqrt{329}}{2} = 124{\cdot}9842500,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 39{\cdot}1383571\,6, \qquad 21+\sqrt{329} = 39{\cdot}1383571,$$

$$xyzw+\frac{1}{xyzw} = 745{\cdot}3363215, \qquad \frac{747+41\sqrt{329}}{2} = 745{\cdot}3363215.$$

The first of the four expressions does not suggest any simple surd; but

$$\left(\frac{xy}{zw}\right)^2+\left(\frac{zw}{xy}\right)^2 = 52562{\cdot}47949, \qquad 26280+1449\sqrt{329} = 52562{\cdot}47951.$$

Assume the truth of the equalities suggested by the first three results; the truth of the fourth suggested equality is then confirmed by the modular equation of order 5; by solving quadratic equations, we then get

$$\left(\frac{xy}{zw}\right)^2 = \tfrac{1}{2}(26280+1449\sqrt{329})+\tfrac{1}{2}(11753\sqrt{5}+648\sqrt{1645})$$

$$= (\sqrt{5}+2)^4\left(\frac{9\sqrt{329}+73\sqrt{5}}{2}\right),$$

$$\frac{xz}{yw} = \left\{\sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)}\right\}^2,$$

$$\frac{xw}{yz} = \tfrac{1}{2}(21+\sqrt{329})+\tfrac{1}{2}(3\sqrt{47}+7\sqrt{7})$$

$$= \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right),$$

$$xyzw = \left\{\sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)}\right\}^2.$$

Hence

$$G_{1645}^2 = (\sqrt{5}+2)\left(\frac{9\sqrt{329}+73\sqrt{5}}{2}\right)^{\frac{1}{4}}\sqrt{\left\{\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)\right\}}\times$$

$$\times\left\{\sqrt{\left(\frac{127+7\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{119+7\sqrt{329}}{8}\right)}\right\}\times$$

$$\times\left\{\sqrt{\left(\frac{751+41\sqrt{329}}{8}\right)}+\sqrt{\left(\frac{743+41\sqrt{329}}{8}\right)}\right\}.$$

(vii) $\mathbf{n = 1677 = 3\times 13\times 43}$. (G. VIII, class-number 2.)

Let

$$G_{1677} = x, \qquad G_{1677/9} = y, \qquad G_{1677/169} = z, \qquad G_{1677/1521} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 2{\cdot}25277\,73312, \qquad \log_{10} y = 0{\cdot}70075\,41111,$$
$$\log_{10} z = 0{\cdot}10384\,39741, \qquad \log_{10} w = 0{\cdot}00020\,64678.$$

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 707{\cdot}10167\,99, \qquad 353+54\sqrt{43} = 707{\cdot}10168\,06,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 45{\cdot}27648\,847,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 28{\cdot}11487\,707, \qquad 15+2\sqrt{43} = 28{\cdot}11487\,70,$$

$$xyzw+\frac{1}{xyzw} = 1141{\cdot}77943\,7.$$

The second of the four expressions does not suggest any simple surd; but

$$\left(\frac{xz}{yw}\right)^2+\left(\frac{yw}{xz}\right)^2 = 2047{\cdot}96040\,8, \qquad 1025+156\sqrt{43} = 2047{\cdot}96041\,0.$$

Assume the truth of the equalities suggested by these three results;* it may then be deduced from the modular equation of order 3 that

$$(xyzw)^3+\frac{1}{(xyzw)^3} = 7442\,44540+1134\,96228\sqrt{43};$$

by solving quadratic equations, we then get

$$\frac{xy}{zw} = \left\{\sqrt{\left(\frac{355+54\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{351+54\sqrt{43}}{4}\right)}\right\}^2,$$

$$\left(\frac{xz}{yw}\right)^2 = \tfrac{1}{2}(1025+156\sqrt{43})+\tfrac{1}{2}(164\sqrt{39}+25\sqrt{1677})$$
$$= \tfrac{1}{2}(25+4\sqrt{39})(41+\sqrt{1677})$$

* In his note-book all that Ramanujan wrote under the heading $\sqrt{1677}$ was merely

$$S+\frac{1}{S} = 15+2\sqrt{43}.$$

If he was using seven-figure tables of logarithms, he may have noticed that

$$\tfrac{1}{2}(1136+175\sqrt{43}) = 1141{\cdot}77537,$$

and consequently have been hindered from progressing in the right direction.

$$= (\sqrt{13}+2\sqrt{3})^2\left(\frac{\sqrt{43}+\sqrt{39}}{2}\right)^2,$$

$$\frac{xw}{yz} = \left\{\sqrt{\left(\frac{17+2\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{13+2\sqrt{43}}{4}\right)}\right\}^2,$$

$$(xyzw)^3 = (3721\,22270+567\,48114\sqrt{43})+$$
$$+(1032\,08148\sqrt{13}+157\,39095\sqrt{559})$$
$$= (4414\sqrt{13}+2427\sqrt{43})(6485\sqrt{13}+23382)$$
$$= (4414\sqrt{13}+2427\sqrt{43})\left(\frac{\sqrt{13}+3}{2}\right)^9.$$

Hence we have the new result

$$G_{1677}^2 = (4414\sqrt{13}+2427\sqrt{43})^{\frac{1}{3}}\sqrt{\left\{\left(\frac{\sqrt{13}+3}{2}\right)^3\left(\frac{\sqrt{43}+\sqrt{39}}{2}\right)(\sqrt{13}+2\sqrt{3})\right\}}\times$$
$$\times\left\{\sqrt{\left(\frac{355+54\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{351+54\sqrt{43}}{4}\right)}\right\}\times$$
$$\times\left\{\sqrt{\left(\frac{17+2\sqrt{43}}{4}\right)}+\sqrt{\left(\frac{13+2\sqrt{43}}{4}\right)}\right\}.$$

(viii) $\mathbf{n = 66 = 2\times 3\times 11}$. (G. IV, class-number 2.)

Let $$g_{66} = x, \qquad g_{66/9} = y.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}38658\,60801, \qquad \log_{10} y = 0{\cdot}07860\,26448.$$

We thus have

$$x^2y^2+\frac{1}{x^2y^2} = 8{\cdot}63616\,8614,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2} = 4{\cdot}37228\,1320, \qquad \frac{3+\sqrt{33}}{2} = 4{\cdot}37228\,132.$$

The first expression does not suggest any simple surd. Assume the truth of the equality suggested by the second result; then, from the modular equation of order 3, namely,

$$8\left(x^6y^6+\frac{1}{x^6y^6}\right) = \frac{x^{12}}{y^{12}}+\frac{y^{12}}{x^{12}}-18,$$

we deduce that

$$x^6y^6+\frac{1}{x^6y^6} = 308+54\sqrt{33},$$

so that

$$x^6y^6 = (154+27\sqrt{33})+(33\sqrt{22}+63\sqrt{6})$$
$$= (\sqrt{3}+\sqrt{2})^3(3\sqrt{11}+7\sqrt{2}).$$

Since it has been assumed that

$$\frac{x}{y} = \sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)},$$

we get at once

$$g_{66}^2 = \sqrt{(\sqrt{3}+\sqrt{2})}.(3\sqrt{11}+7\sqrt{2})^{\frac{1}{4}}\left\{\sqrt{\left(\frac{7+\sqrt{33}}{8}\right)} + \sqrt{\left(\frac{\sqrt{33}-1}{8}\right)}\right\}.$$

(ix) $\mathbf{n = 114 = 2\times 3\times 19}$. (G. IV, class-number 2.)

Let $g_{114} = x, \qquad g_{114/9} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}53172\,38801, \qquad \log_{10} y = 0{\cdot}12706\,35748.$$

We thus have

$$x^2y^2 + \frac{1}{x^2y^2} = 20{\cdot}82475\,164, \qquad \frac{19+3\sqrt{57}}{2} = 20{\cdot}82475\,16,$$

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} = 6{\cdot}60157\,3953.$$

The second expression does not suggest any simple surd. Assume the truth of the equality suggested by the first result; then, from the modular equation of order 3, quoted in (viii), we deduce that

$$\frac{x^6}{y^6} + \frac{y^6}{x^6} = 18\sqrt{57}+132,$$

so that

$$x^6/y^6 = (9\sqrt{57}+66)+(11\sqrt{38}+27\sqrt{6})$$
$$= (\sqrt{3}+\sqrt{2})^3(19+3\sqrt{2}).$$

Since it has been assumed that

$$xy = \sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)} + \sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)},$$

we get at once

$$g_{114}^2 = \sqrt{(\sqrt{3}+\sqrt{2})}.(19+3\sqrt{2})^{\frac{1}{4}}\left\{\sqrt{\left(\frac{23+3\sqrt{57}}{8}\right)} + \sqrt{\left(\frac{15+3\sqrt{57}}{8}\right)}\right\}.$$

(x) $\mathbf{n = 126 = 2\times 3^2\times 7}$. (G. IV, class-number 2.)

Let $g_{126} = x, \qquad g_{126/49} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}56287\,11093, \qquad \log_{10} y = 0{\cdot}01307\,66014,$$

and hence

$$x^6y^6 + \frac{1}{x^6y^6} = 2855{\cdot}527307, \qquad 1430+1008\sqrt{2} = 2855{\cdot}52727,$$

$$\frac{x}{y} + \frac{y}{x} = 3{\cdot}82842\,7124, \qquad 1+2\sqrt{2} = 3{\cdot}82842\,712.$$

Assume the truth of the equalities suggested by these results; the modular equation of order 7, namely,

$$\frac{x^4}{y^4}+\frac{y^4}{x^4}-7 = 2\sqrt{2}\left(x^3y^3+\frac{1}{x^3y^3}\right)$$

is then satisfied, and

$$\begin{aligned} x^3y^3 &= (14+9\sqrt{2})+(3\sqrt{21}+2\sqrt{42}) \\ &= (\sqrt{7}+\sqrt{6})\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3. \end{aligned}$$

Since it has been assumed that

$$\begin{aligned} \frac{x}{y} &= \left\{\sqrt{\left(\frac{3+2\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{2\sqrt{2}-1}{4}\right)}\right\}^2 \\ &= \left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^4, \end{aligned}$$

we get at once

$$g_{126} = (\sqrt{7}+\sqrt{6})^{\frac{1}{2}}\sqrt{\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)}\left\{\sqrt{\left(\frac{3+\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{\sqrt{2}-1}{4}\right)}\right\}^2.$$

(xi) $\mathbf{n = 138 = 2\times 3\times 23}$. (G. IV, class-number 2.)

Let $\qquad g_{138} = x, \qquad g_{138/9} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10}x = 0{\cdot}59256\,72130, \qquad \log_{10}y = 0{\cdot}14734\,87651.$$

We thus have

$$x^2y^2+\frac{1}{x^2y^2} = 30{\cdot}22096\,014,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2} = 7{\cdot}89897\,9486, \qquad 3+2\sqrt{6} = 7{\cdot}89897\,949.$$

The first expression does not suggest any simple surd. Assume the truth of the equality suggested by the second result; then, from the modular equation of order 3, quoted in (viii), we deduce that

$$x^6y^6+\frac{1}{x^6y^6} = 13754+5616\sqrt{6}$$

so that

$$\begin{aligned} x^6y^6 &= (6877+2808\sqrt{6})+(1014\sqrt{46}+828\sqrt{69}) \\ &= (78\sqrt{2}+23\sqrt{23})\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^3. \end{aligned}$$

Since it has been assumed that

$$\frac{x}{y} = \sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)},$$

we get at once

$$g_{138}^2 = (78\sqrt{2}+23\sqrt{23})^{\frac{1}{2}}\sqrt{\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)}\left\{\sqrt{\left(\frac{5+2\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{6}}{4}\right)}\right\}.$$

(xii) $\mathbf{n = 154 = 2\times 7\times 11}$. (G. IV, class-number 2.)

Let $g_{154} = x, \qquad g_{154/49} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}63022\,02211, \qquad \log_{10} y = 0{\cdot}02386\,60604.$$

We thus have

$$x^2y^2+\frac{1}{x^2y^2} = 20{\cdot}38083\,151, \qquad 11+2\sqrt{22} = 20{\cdot}38083\,152,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2} = 16{\cdot}38083\,153, \qquad 7+2\sqrt{22} = 16{\cdot}38083\,152.$$

Assume the truth of the equalities suggested by these results; by solving quadratic equations, we then get

$$x^2y^2 = \left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)}+\sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\}^2,$$

$$\frac{x^2}{y^2} = \frac{(7+2\sqrt{22})+(2\sqrt{14}+\sqrt{77})}{2}$$

$$= (2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right).$$

Hence

$$g_{154}^2 = \sqrt{\left\{(2\sqrt{2}+\sqrt{7})\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)\right\}}\left\{\sqrt{\left(\frac{13+2\sqrt{22}}{4}\right)}+\sqrt{\left(\frac{9+2\sqrt{22}}{4}\right)}\right\}.$$

(xiii) $\mathbf{n = 238 = 2\times 7\times 17}$. (G. IV, class-number 2.)

Let $g_{238} = x, \qquad g_{238/49} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}80176\,63854, \qquad \log_{10} y = 0{\cdot}04960\,40064.$$

We thus have

$$x^2y^2+\frac{1}{x^2y^2} = 50{\cdot}45584\,420, \qquad 25+18\sqrt{2} = 50{\cdot}4558441,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2} = 31{\cdot}97056\,269, \qquad 15+12\sqrt{2} = 31{\cdot}9705627.$$

Assume the truth of the equalities suggested by these results; we then get

$$xy+\frac{1}{xy} = 3+3\sqrt{2}, \qquad \frac{x}{y}+\frac{y}{x} = 3+2\sqrt{2},$$

and consequently, by solving quadratic equations,

$$xy = \left\{\sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)}\right\}^2,$$

$$\frac{x}{y} = \left\{\sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)}\right\}^2.$$

Hence

$$g_{238} = \left\{\sqrt{\left(\frac{5+3\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{1+3\sqrt{2}}{4}\right)}\right\}\left\{\sqrt{\left(\frac{5+2\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{1+2\sqrt{2}}{4}\right)}\right\}.$$

(xiv) $\mathbf{n = 310 = 2\times 5\times 31}$. (G. IV, class-number 2.)

Let $\qquad g_{310} = x, \qquad g_{310/25} = y.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 0{\cdot}92567\,26924, \qquad \log_{10} y = 0{\cdot}12492\,17266.$$

We thus have

$$x^2y^2+\frac{1}{x^2y^2} = 126{\cdot}24555\,32, \qquad 63+20\sqrt{10} = 126{\cdot}24555\,32,$$

$$\frac{x^2}{y^2}+\frac{y^2}{x^2} = 39{\cdot}97366\,597, \qquad 21+6\sqrt{10} = 39{\cdot}97366\,60.$$

Assume the truth of the equalities suggested by these results; we then get

$$xy+\frac{1}{xy} = 5+2\sqrt{10}, \qquad \frac{x}{y}+\frac{y}{x} = 3\sqrt{2}+\sqrt{5},$$

and consequently, by solving quadratic equations,

$$xy = \left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\}^2,$$

$$\frac{x}{y} = \frac{(3\sqrt{2}+\sqrt{5})+(3+\sqrt{10})}{2}$$

$$= \left(\frac{3+\sqrt{5}}{2}\right)(\sqrt{2}+1).$$

Hence

$$g_{310} = \left(\frac{\sqrt{5}+1}{2}\right)\sqrt{(\sqrt{2}+1)}\left\{\sqrt{\left(\frac{7+2\sqrt{10}}{4}\right)}+\sqrt{\left(\frac{3+2\sqrt{10}}{4}\right)}\right\}.$$

(xv) $\mathbf{n = 522 = 2\times 3^2\times 29}$. (G. IV, class-number 2.)

Let $\qquad g_{522} = x, \qquad g_{522/9} = y, \qquad g_{522/81} = z.$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 1{\cdot}22359\,00942, \qquad \log_{10} z = 0{\cdot}06890\,95299,$$

$$\log_{10} y = 0{\cdot}35769\,16991.$$

We thus have

$$\frac{x}{z}+\frac{z}{x} = 14{\cdot}34846\,926, \qquad 7+3\sqrt{6} = 14{\cdot}34846\,923,$$

$$y^4+\frac{1}{y^4} = 27{\cdot}00000\,011,$$

$$\left(\frac{xz}{y^2}\right)+\frac{y^2}{xz} = 4{\cdot}04151\,1646.$$

The last expression does not suggest a simple surd; but

$$\left(\frac{xz}{y^2}\right)^3+\left(\frac{y^2}{xz}\right)^3 = 53{\cdot}88877\,419, \qquad 22\sqrt{6} = 53{\cdot}88877\,43.$$

Assume the truth of the equalities suggested by these results;* by solving quadratic equations, we then get

$$\frac{x}{z} = \left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}^2, \qquad y^2 = \frac{\sqrt{29}+5}{2},$$

$$\left(\frac{xz}{y^2}\right)^3 = 11\sqrt{6}+5\sqrt{29}.$$

Hence

$$g_{522} = (11\sqrt{6}+5\sqrt{29})^{\frac{1}{3}}\sqrt{\left(\frac{\sqrt{29}+5}{2}\right)}\left\{\sqrt{\left(\frac{9+3\sqrt{6}}{4}\right)}+\sqrt{\left(\frac{5+3\sqrt{6}}{4}\right)}\right\}.$$

(xvi) $\mathbf{n = 630 = 2\times 3^2\times 5\times 7}$. (G. VIII, class-number 2.)

Let

$$g_{630} = x, \qquad g_{630/25} = y, \qquad g_{630/49} = z, \qquad g_{630/1225} = 1/g_{630/81} = w.$$

Then, by direct computation of the $q$-products,

$$\log_{10} x = 1{\cdot}35164\,14470, \qquad \log_{10} y = 0{\cdot}21012\,22291,$$
$$\log_{10} z = 0{\cdot}12857\,96417, \qquad \log_{10} w = \bar{1}{\cdot}91678\,12097.$$

We thus have

$$\frac{xy}{zw}+\frac{zw}{xy} = 32{\cdot}87042\,621, \qquad \frac{35+3\sqrt{105}}{2} = 32{\cdot}87042\,61,$$

$$\frac{xz}{yw}+\frac{yw}{xz} = 22{\cdot}60321\,445,$$

$$\frac{xw}{yz}+\frac{yz}{xw} = 8{\cdot}62347\,5369, \qquad \frac{7+\sqrt{105}}{2} = 8{\cdot}62347\,538,$$

$$xyzw+\frac{1}{xyzw} = 40{\cdot}49390\,162, \qquad 20+2\sqrt{105} = 40{\cdot}49390\,15.$$

* The value assumed for $g_{58}$ agrees with the value given in Weber's *Algebra*.

The second result does not suggest a simple surd; but

$$\left(\frac{xz}{yw}\right)^3+\left(\frac{yw}{xz}\right)^3 = 11480{\cdot}29249, \qquad 5742+560\sqrt{105} = 11480{\cdot}29245.$$

Assume the truth of the equalities suggested by these results; by solving quadratic equations, we then get

$$\frac{xy}{zw} = \frac{(35+3\sqrt{105})+(15\sqrt{5}+7\sqrt{21})}{4}$$

$$= \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^2\left(\frac{3+\sqrt{5}}{2}\right)^2,$$

$$\left(\frac{xz}{yw}\right)^{\frac{3}{2}} = (5\sqrt{30}+7\sqrt{14})+(7\sqrt{15}+10\sqrt{7})$$

$$= (\sqrt{15}+\sqrt{14})(\sqrt{2}+1)^3,$$

$$\sqrt{\left(\frac{xw}{yz}\right)}+\sqrt{\left(\frac{yz}{xw}\right)} = \frac{\sqrt{15}+\sqrt{7}}{2}, \qquad \sqrt{(xyzw)}+\frac{1}{\sqrt{(xyzw)}} = \sqrt{15}+\sqrt{7}.$$

Hence

$$g_{630} = (\sqrt{15}+\sqrt{14})^{\frac{1}{4}}\sqrt{\left\{\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)(\sqrt{2}+1)\right\}}\times$$

$$\times\left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+4}{8}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-4}{8}\right)}\right\}\times$$

$$\times\left\{\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}+2}{4}\right)}+\sqrt{\left(\frac{\sqrt{15}+\sqrt{7}-2}{4}\right)}\right\}.$$

The last two factors may be written in the more compact form

$$\sqrt{\left[\left\{\sqrt{\left(\frac{11+\sqrt{105}}{8}\right)}+\sqrt{\left(\frac{3+\sqrt{105}}{8}\right)}\right\}\left\{\sqrt{\left(\frac{11+\sqrt{105}}{2}\right)}+\sqrt{\left(\frac{9+\sqrt{105}}{2}\right)}\right\}\right]},$$

so as to resemble the factors occurring in other singular moduli of this paper.

# THE RATIONALITY OF A CERTAIN ALGEBRAIC INTEGRAL

*By* A. E. JOLLIFFE (*London*)

[Received 26 February 1932]

**1.** It is a well-known result, often used as an examination question, that the necessary and sufficient condition that the integral

$$\int \frac{a'x^2+2h'x+b'}{(ax^2+2hx+b)^2}\,dx$$

should be rational is $ab'+a'b-2hh'=0$, that is, that the mutual invariant of the two quadratic forms should vanish, or that the two forms should be apolar.

This result has been generalized by M. Hanna in *Proc. London Math. Soc.*, vol. 33, p. 190, in the form that the integral

$$\int \frac{U_{2n}\,dx}{(U_2)^{n+1}},$$

where $U_{2n}$ and $U_2$ are polynomials of degree $2n$ and 2, is rational if and only if $U_{2n}$ and $(U_2)^n$ are apolar.

This result has been known to me and doubtless to others for a long time; but I can give no reference to any place where it may be found, so that it is very possible that it appears in print for the first time in M. Hanna's paper.

I propose here to give a very short and simple proof of this result and also to give the answer to this question which arises in connexion with it:

If $\int \frac{U_{2n}\,dx}{(U_2)^{n+1}}$ is rational, it is of the form $V_{2n}/(U_2)^n$. The quantic $V_{2n}(x,y)$ must be a joint covariant of $U_{2n}(x,y)$ and $U_2(x,y)$, and must therefore be expressible by means of direct operations involving $U_{2n}$ and $U_2$. What, then, is this expression?

**2.** If we write $x/y$ for $x$, the integral becomes

$$\int \frac{U_{2n}(x,y)}{\{U_2(x,y)\}^{n+1}}(y\,dx - x\,dy).$$



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If $U_2(x,y) \equiv b_0x^2+2b_1xy+b_2y^2 \equiv (lx+my)(l'x+m'y)$, and we write $lx+my = X$, $l'x+m'y = Y$, the integral takes the form

$$\int \frac{A_0X^{2n}+A_1X^{2n-1}Y+\ldots+A_{2n}Y^{2n}}{(lm'-l'm)X^{n+1}Y^{n+1}}(Y\,dX - X\,dY),$$

which becomes, when we put $X = Yz$,

$$\int \frac{A_0z^{2n}+A_1z^{2n-1}+\ldots+A_{2n}}{(lm'-l'm)z^{n+1}}\,dz,$$

which is rational if and only if $A_n$ vanishes, that is, if

$$\left(\frac{\partial^2}{\partial X\partial Y}\right)^n(A_0X^{2n}+\ldots+A_{2n}Y^{2n}) = 0,$$

that is, if the joint invariant of $U_2^n$ and $U_{2n}$ vanishes, or $U_2^n$ and $U_{2n}$ are apolar.

**3.** The actual integral is

$$\frac{1}{(lm'-l'm)}\left\{\frac{A_0z^n}{n}+\frac{A_1z^{n-1}}{n-1}+\ldots+A_{n-1}z-\frac{A_{n+1}}{z}-\ldots-\frac{A_{2n}}{nz^n}\right\},$$

i.e.
$$\frac{1}{(lm'-l'm)X^nY^n}\left\{\frac{A_0X^{2n}}{n}+\frac{A_1X^{2n-1}Y}{n-1}+\ldots-\frac{A_{2n}Y^{2n}}{n}\right\}.$$

The Jacobian of

$$A_0X^{2n}+A_1X^{2n-1}Y+\ldots+A_{2n}Y^{2n}$$

and $XY$ is

$$J \equiv 2\{nA_0X^{2n}+(n-1)A_1X^{2n-1}Y+\ldots+A_{n-1}X^{n+1}Y^{n-1}-$$
$$-A_{n+1}X^{n-1}Y^{n+1}-\ldots-nA_{2n}Y^{2n}\}.$$

The result of operating on $J$ with

$$\frac{1}{n^2}+\frac{XYD}{n^2(n-1)^2}+\frac{X^2Y^2D^2}{n^2(n-1)^2(n-2)^2}+\ldots,$$

where $D$ denotes the operator $\dfrac{\partial^2}{\partial X\partial Y}$, is

$$2\left\{\frac{A_1}{n}X^{2n}+\frac{A_1}{n-1}X^{2n-1}Y+\ldots-\frac{A_{2n}}{n}Y^{2n}\right\}.$$

This follows from the identity

$$\frac{1}{n^2}+\frac{(2n-r)r}{n^2(n-1)^2}+\frac{(2n-r)(2n-r-1)r(r-1)}{n^2(n-1)^2(n-2)^2}+\ldots = \frac{1}{(n-r)^2},$$

if $r$ is a positive integer less than $n$, the proof of which by induction is immediate.

Hence $V_{2n}(x, y)$ is

$$\frac{1}{8(b_1^2-b_0b_2)}\left\{\frac{1}{n^2}+\frac{U_2D}{n^2(n-1)^2}+\frac{U_2^2D^2}{n^2(n-1)^2(n-2)^2}+\cdots\right\}J,$$

where $D$ denotes

$$\frac{1}{4(b_0b_2-b_1^2)}\left(b_2\frac{\partial^2}{\partial x^2}-2b_1\frac{\partial^2}{\partial x\partial y}+b_0\frac{\partial^2}{\partial y^2}\right)$$

and $J$ is the Jacobian of $U_{2n}(x, y)$ and $U_2(x, y)$.

If $U_{2n}(x, y)$ is taken with binomial coefficients, the conventional Jacobian of $U_{2n}(x, y)$ and $U_2(x, y)$ is the $J$ above divided by $4n$.

# ON HILBERT TRANSFORMS

*By* G. H. HARDY (*Cambridge*)

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**1.** Two functions $f(x)$ and $g(x)$ of the real variable $x$ may be said to be *conjugate*, or to be *Hilbert transforms* of one another, if they are connected by the formulae

$$g(x) \equiv -\frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty} f(t)\log\left|1-\frac{x}{t}\right| dt, \tag{1.1.1}$$

$$f(x) \equiv \frac{1}{\pi}\frac{d}{dx}\int_{-\infty}^{\infty} g(t)\log\left|1-\frac{x}{t}\right| dt; \tag{1.1.2}$$

or by

$$g(x) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt, \tag{1.2.1}$$

$$f(x) \equiv -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t-x}\, dt. \tag{1.2.2}$$

Here '$\equiv$' denotes equivalence, i.e. equality for almost all $x$, and the integrals (1.2) are 'principal values' in the sense of Cauchy. The relation is 'real', but is a derivative of that between the conjugate components of an analytic function of the complex variable.

In two notes published in 1924 in the *Messenger of Mathematics*,† I set out to give a direct and self-contained theory of Hilbert transforms of the class $L^2$.‡ The theory may be based on the theory of analytic functions, on the theory of Fourier transforms, or on reciprocal relations between infinite series;§ but it is interesting to develop it independently, and Watson‖ has shown that the method which I used can be applied to more general problems.

There is, however, a mistake in my analysis, which was pointed out to me by Professor Watson.†† This is not difficult to rectify; but

† Hardy (**1**, **2**). In these notes $g(x)$ is defined with the opposite sign. According to (1.1) and (1.2), the conjugate of $\cos mx$ is $-\sin mx$ if $m > 0$, $\sin mx$ if $m < 0$.

‡ The class of measurable functions whose squares are integrable over $(-\infty, \infty)$.

§ See M. Riesz (**5**, **6**), Titchmarsh (**7**, **8**, **9**). Riesz and Titchmarsh deal also with the more difficult case in which $f$ and $g$ belong to a general Lebesgue class $L^p$ with $p > 1$.

‖ Watson (**10**).

†† In **1**, p. 26, § 4. From 'In order to prove this ...' to the end of § 4 is nonsense.

while reconsidering the proof I have found a number of places where the analysis can be simplified and shortened materially. I therefore restate the whole argument in §§ 2–10 of the present note. The proofs of Lemmas 2–4 are substantially the same as before, but the rest of the argument has been entirely recast. In § 11, I indicate shortly the lines of an alternative, and in some ways still simpler, proof of Theorem 1, suggested to me by Professor Titchmarsh.

**2.** The theorems to be proved are as follows:

THEOREM 1. *If $f(x)$ is $L^2$, then there is a function $g(x)$ which has the properties*:

(*a*) *$g(x)$ is $L^2$,*

$$(b)\quad \int_{-\infty}^{\infty} g^2(x)\,dx = \int_{-\infty}^{\infty} f^2(x)\,dx, \tag{2.1}$$

(*c*) *$f(x)$ and $g(x)$ satisfy* (1.1).

THEOREM 2. *The functions $f(x)$ and $g(x)$ also satisfy* (1.2), *the integrals being Lebesgue integrals at infinity, but principal values about $t = x$.*

It is to be observed that both the statement and the proof of Theorem 1 are independent of Theorem 2, although it is in the form (1.2) that the relation has usually been stated. Theorem 1 corresponds to Plancherel's fundamental theorem concerning Fourier transforms, in which the cosine transform of $f(x)$ is defined by

$$g(x) \equiv \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^{\infty} f(t) \frac{\sin xt}{t}\,dt.$$

Here we cannot usually differentiate under the integral sign. Theorem 2 asserts that, in the formulae for Hilbert transforms, we can; and this is one of the most interesting properties of the transformation.

## Lemmas for Theorem 1

**3.** In what follows all numbers which occur are real, and

$$lx = \log|x|.$$

LEMMA 1. *Suppose that $\chi(t) = \chi(t, h, h', \ldots)$ has the following properties*:

(*a*) *$\chi(t)$ belongs to $L^2$ for all values of $h, h', \ldots$ in question*;

$$(b)\quad \int_{-\infty}^{\infty} \chi^2(t, h, h', \ldots)\,dt < A,$$

*where $A$ is independent of $h, h', \ldots$*;

(c) $$\chi(t,h,h',\ldots)\to\chi^*(t),$$

*when $h,h',\ldots$ tend to certain limiting values, for almost all $t$. Then $f(t)\chi^*(t)$ belongs to $L$,† and*

$$\int_{-\infty}^{\infty}\chi(t,h,h',\ldots)f(t)\,dt\to\int_{-\infty}^{\infty}\chi^*(t)f(t)\,dt,$$

*for every $f(t)$ of $L^2$.‡*

LEMMA 2. *If*

$$I(a,a',b,b')=\int_{-\infty}^{\infty}l\left(\frac{x-a}{x-a'}\right)l\left(\frac{x-b}{x-b'}\right)dx, \tag{3.1}$$

*then*
$$I(a,a',b,b')=\tfrac{1}{2}\pi^2(|a-b'|+|a'-b|-|a-b|-|a'-b'|). \tag{3.2}$$

*In particular, if $h$ and $h'$ are positive, and*

$$J(t,u,h,h')=I(t-h,t+h,u-h',u+h')$$
$$=\int_{-\infty}^{\infty}l\left(\frac{t-x+h}{t-x-h}\right)l\left(\frac{u-x+h'}{u-x-h'}\right)dx, \tag{3.3}$$

*then*
$$J(t,u,h,h')=\pi^2\mu(t-u,h,h'), \tag{3.4}$$

*where*
$$\mu(w,h,h')=0 \qquad (|w|\geqslant h+h'), \tag{3.5.1}$$
$$\mu(w,h,h')=\min(h+h'-|w|,h+h'-|h-h'|) \qquad (|w|\leqslant h+h'). \tag{3.5.2}$$

The proof of this lemma is a matter of elementary calculation. Since
$$2l\left(\frac{x-a}{x-a'}\right)l\left(\frac{x-b}{x-b'}\right)$$
$$=\left\{l\left(\frac{x-a}{x-b'}\right)\right\}^2+\left\{l\left(\frac{x-a'}{x-b}\right)\right\}^2-\left\{l\left(\frac{x-a}{x-b}\right)\right\}^2-\left\{l\left(\frac{x-a'}{x-b'}\right)\right\}^2,$$
the general formula (3.2) will follow from the particular case

$$I(\alpha,\beta,\alpha,\beta)=\int_{-\infty}^{\infty}\left\{l\left(\frac{x-\alpha}{x-\beta}\right)\right\}^2dx=\pi^2|\alpha-\beta|.$$

And if we suppose, for example, that $\alpha>\beta$, then simple transformations give

$$I(\alpha,\beta,\alpha,\beta)=(\alpha-\beta)\int_{-\infty}^{\infty}\frac{(lu)^2}{(u-1)^2}\,du=2(\alpha-\beta)\int_{-1}^{1}\frac{(lu)^2}{(1-u)^2}\,du$$

† Is integrable over $(-\infty,\infty)$.

‡ See Young (**11**), and the writings of Lebesgue there referred to.

$$= 2(\alpha-\beta)\int_{-1}^{1}(lu)^2(1+2u+3u^2+\ldots)\,du$$

$$= 8(\alpha-\beta)\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right) = \pi^2(\alpha-\beta).$$

This proves (3.2), and (3.4) is a special case. Geometrically, $\mu$ is an even non-negative function of $w$ which has the constant value $2h$ or $2h'$ (whichever is smaller) from $w=0$ to $w=|h-h'|$, decreases linearly until it vanishes for $w=h+h'$, and then remains zero. If $h \leqslant h'$,

$$\frac{1}{4hh'}\int_{-\infty}^{\infty}\mu\,dw = \frac{1}{2hh'}\int_{0}^{h'-h}2h\,dw + \frac{1}{2hh'}\int_{h'-h}^{h+h'}(h+h'-w)\,dw$$

$$= \frac{h'-h}{h'}+\frac{h}{h'} = 1; \tag{3.6}$$

and this is true by symmetry for all positive $h$, $h'$.

4. Lemma 3. *If $f$ is $L^2$, and*

$$\chi(t,h,h') = \frac{1}{4hh'}\int_{-\infty}^{\infty}f(t+w)\mu(w,h,h')\,dw,$$

*then*

$$\int_{-\infty}^{\infty}\chi^2(t,h,h')\,dt \leqslant 2\int_{-\infty}^{\infty}f^2(t)\,dt.$$

We may suppose $h \leqslant h'$. Then, using Schwarz's inequality, and observing that $\mu = 0$ if $|w| > h+h'$, we find

$$\chi^2 \leqslant \frac{1}{16h^2h'^2}\int_{-h-h'}^{h+h'}f^2(t+w)\,dw\int_{-\infty}^{\infty}\mu^2\,dw.$$

Since

$$\int_{-\infty}^{\infty}\mu^2\,dw \leqslant 2h\int_{-\infty}^{\infty}\mu\,dw = 8h^2h',$$

we deduce

$$\int_{-\infty}^{\infty}\chi^2\,dt \leqslant \frac{1}{2h'}\int_{-\infty}^{\infty}dt\int_{-h-h'}^{h+h'}f^2(t+w)\,dw$$

$$= \frac{1}{2h'}\int_{-h-h'}^{h+h'}dw\int_{-\infty}^{\infty}f^2(t+w)\,dw \leqslant 2\int_{-\infty}^{\infty}f^2\,dt.$$

LEMMA 4. *If $h$ and $h'$ tend to zero, then*

$$\chi(t,h,h') \to f(t)$$

*for almost all $t$.*

We may consider separately systems $(h,h')$ for which $h \leqslant h'$ and $h' \leqslant h$. Suppose, then, that $h \leqslant h'$, and let

$$\psi(w) = \psi(w,t) = \tfrac{1}{2}\{f(t+w)+f(t-w)-2f(t)\}. \tag{4.1}$$

After (3.6), we have

$$\chi(t,h,h')-f(t) = \frac{1}{2hh'}\int_0^{\infty} \psi(w)\mu(w)\,dw$$

$$= \frac{1}{h'}\int_0^{h'-h} \psi(w)\,dw + \frac{1}{2hh'}\int_{h'-h}^{h+h'} \psi(w)(h+h'-w)\,dw,$$

$$|\chi(t,h,h')-f(t)| \leqslant \frac{1}{h'}\int_0^{2h'} |\psi(w)|\,dw,$$

which tends to zero for almost all $t$.

It follows from Lemmas 3 and 4 that $\chi(t)$ satisfies the conditions of Lemma 1, with $\chi^* = f$.

## Proof of Theorem 1

**5.** We suppose first that $f(t) = 0$ for $|t| > T$.† We write

$$\phi(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,l\left(1-\frac{x}{t}\right)dt; \tag{5.1}$$

the integral (in which the infinite limits are at present apparent only) is convergent because $f$ is $L^2$, and uniformly convergent in any finite interval of $x$. Further, we write

$$\Delta = \Delta(x,h) = \frac{\phi(x+h)-\phi(x-h)}{2h} = \frac{1}{2\pi h}\int_{-\infty}^{\infty} f(t)\,l\left(\frac{t-x+h}{t-x-h}\right)dt, \tag{5.2}$$

$$I(h,h') = \int_{-\infty}^{\infty} \Delta(x,h)\Delta(x,h')\,dx = \int_{-\infty}^{\infty} \Delta\Delta'\,dx. \tag{5.3}$$

Then

$$I(h,h') = \frac{1}{4\pi^2 hh'}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(t)\,l\left(\frac{t-x+h}{t-x-h}\right)dt \int_{-\infty}^{\infty} f(u)\,l\left(\frac{u-x+h'}{u-x-h'}\right)du.$$

† The mistake in (**1**) arose from an attempt to avoid this distinction.

The triple integral is absolutely convergent, since $f$ vanishes for large $t$ or $u$, and therefore

$$I(h,h') = \frac{1}{4\pi^2 hh'}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} f(u)\,du \int_{-\infty}^{\infty} l\left(\frac{t-x+h}{t-x-h}\right) l\left(\frac{u-x+h'}{u-x-h'}\right) dx$$

$$= \frac{1}{4hh'}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} f(u)\,\mu(t-u,h,h')\,du$$

$$= \frac{1}{4hh'}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} f(t+w)\,\mu(w,h,h')\,dw = \int_{-\infty}^{\infty} f(t)\,\chi(t,h,h')\,dt.$$

Since $\chi$ satisfies the conditions of Lemma 1, it follows that

$$I(h,h') \to \int_{-\infty}^{\infty} f^2(t)\,dt,$$

and hence that

$$\int_{-\infty}^{\infty} (\Delta-\Delta')^2\,dx = I(h,h)-2I(h,h')+I(h',h') \to 0. \tag{5.4}$$

6. We can now apply the classical theory of mean-square convergence. It follows from (5.4) that $\Delta(x,h)$ converges in mean square to a function $g(x)$ of $L^2$, and that

$$\int_{-\infty}^{\infty} g^2(x)\,dx = \lim \int_{-\infty}^{\infty} \Delta^2\,dx = \int_{-\infty}^{\infty} f^2(t)\,dt.$$

This is (2.1), for an $f$ which vanishes for $|t| > T$.

Also
$$\int_{-\infty}^{\infty} (\Delta-g)k\,dx \to 0,$$

for any $k$ of $L^2$. If in particular we suppose that $k = 1$ when $x$ lies between 0 and $y$, and $k = 0$ otherwise, we find

$$\int_0^y (\Delta-g)\,dx \to 0$$

and so

$$\int_0^y g\,dx = \lim \int_0^y \Delta\,dx = \lim \frac{1}{2h}\int_0^y \{\phi(x+h)-\phi(x-h)\}\,dx$$

$$= \lim\left\{\frac{1}{2h}\int_{y-h}^{y+h} \phi(u)\,du - \frac{1}{2h}\int_{-h}^{h} \phi(u)\,du\right\} = \phi(y)-\phi(0),$$

since $\phi$ is continuous. That is to say, $\phi$ is the integral of $g$, and $g$ is defined in terms of $f$ by (1.1.1). We call $g$ the *transform* of $f$.

We have thus proved Theorem 1, for $f$ which vanish for large $t$, except for the reciprocity asserted by (1.1.2). This it is more convenient to defer for a moment.

**7.** We now suppose that $f$ is any function of $L^2$, that $f_T$ is $f$ in $(-T, T)$ and 0 outside, and that

$$g_T \equiv -\frac{1}{\pi}\frac{d}{dx}\int_{-T}^{T} f(t)\, l\left(1-\frac{x}{t}\right) dt$$

is the transform of $f_T$.

If $-X \leqslant x \leqslant X$ and $T > X$, then

$$-\frac{d}{dx}\int_{T}^{\infty} f(t)\, l\left(1-\frac{x}{t}\right) dt = \int_{T}^{\infty} \frac{f(t)}{t-x}\, dt$$

exists and is continuous, since the second integral is uniformly convergent; and the same things are true of the integrals over $(-\infty, -T)$. It follows that $g(x)$, defined by (1.1.1), exists for almost all $x$ of $(-X, X)$, and so for almost all $x$; that $\phi$, defined by (5.1), is the integral of $g$; and that $g_T \to g$, when $T \to \infty$, uniformly in any finite interval of $x$.†

If $T' > T$, then $g_{T'} - g_T$ is the transform of the function which is $f$ when $T \leqslant |t| \leqslant T'$ and zero otherwise; and so

$$\int_{-\infty}^{\infty} (g_{T'} - g_T)^2\, dx = \left(\int_{-T'}^{-T} + \int_{T}^{T'}\right) f^2\, dt \to 0$$

when $T$ and $T'$ tend to infinity. Hence $g_T$ tends in mean square to a function $g^*$ when $T \to \infty$, and

$$\int_{-\infty}^{\infty} g^{*2}\, dx = \lim \int_{-\infty}^{\infty} g_T^2\, dx = \lim \int_{-T}^{T} f^2\, dt = \int_{-\infty}^{\infty} f^2\, dt.$$

Finally, as in § 6,

$$\int_{0}^{y} (g_T - g^*)\, dx \to 0,$$

$$\int_{0}^{y} g^*\, dx = \lim \int_{0}^{y} g_T\, dx = \int_{0}^{y} g\, dx$$

† Strictly, $g_T$ and $g$ are equivalent to functions $h_T$ and $h$, and $h_T \to h$ uniformly.

(since $g_T \to g$ uniformly in any finite interval), and so $g^* \equiv g$. This completes the proof of Theorem 1, except for (1.1.2).

8. If $f$ and $f_1$ are two functions of $L^2$, $g$ and $g_1$ their transforms, then, by (2.1),

$$\int_{-\infty}^{\infty} (f \pm f_1)^2 \, dt = \int_{-\infty}^{\infty} (g \pm g_1)^2 \, dt;$$

and so, by subtraction,

$$\int_{-\infty}^{\infty} f f_1 \, dt = \int_{-\infty}^{\infty} g g_1 \, dt. \tag{8.1}$$

If in particular $f_1$ is $\pi$ in $(0, y)$ and 0 outside, then

$$g_1 = -\frac{d}{dx} \int_0^y l\left(1 - \frac{x}{t}\right) dt = -\frac{d}{dx} \int_{-x}^{y-x} lw \, dw = l\left(1 - \frac{y}{x}\right).$$

Taking this $f_1$ and $g_1$ in (8.1), we obtain

$$\pi \int_0^y f(t) \, dt = \int_{-\infty}^{\infty} g(t) \, l\left(1 - \frac{y}{t}\right) dt,$$

and (1.1.2) follows by differentiation. This completes the proof.

## An auxiliary theorem

9. The proof of Theorem 2 depends on another theorem of some intrinsic interest.

THEOREM 3. *If $f$ belongs to $L^2$, then*

$$L(x, \epsilon) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{f(t)}{t - x} \, l\left(\frac{t - x + \epsilon}{t - x - \epsilon}\right) dt \to f(x), \tag{9.1}$$

*when $\epsilon \to 0$, for almost all $x$ and in particular at any point of continuity of $f$.*

The assertion is equivalent to

$$M(\epsilon) = \int_0^{\infty} \frac{\psi(t)}{t} \, l\left(\frac{t + \epsilon}{t - \epsilon}\right) dt \to 0, \tag{9.2}$$

where $\psi(t) = \psi(t, x)$ is defined as in (4.1); and it is sufficient to prove that (9.2) is true when

$$\Psi(t) = \int_0^t \psi^2(u) \, du = o(t) \tag{9.3}$$

for small $t$ (this being true for almost all $x$ when $f$ is $L^2$).

We write $$M(\epsilon) = \int_0^{2\epsilon} + \int_{2\epsilon}^{\infty} = M_1(\epsilon) + M_2(\epsilon).$$

Then

$$M_1^2(\epsilon) \leqslant \int_0^{2\epsilon} \psi^2(t)\,dt \int_{-\infty}^{\infty} \left\{l\left(\frac{t+\epsilon}{t-\epsilon}\right)\right\}^2 \frac{dt}{t^2}$$

$$= \frac{1}{\epsilon} \int_0^{2\epsilon} \psi^2(t)\,dt \int_{-\infty}^{\infty} \left\{l\left(\frac{w+1}{w-1}\right)\right\}^2 \frac{dw}{w^2} \to 0,$$

by (9.3). Also

$$M_2^2(\epsilon) \leqslant \int_{2\epsilon}^{\infty} \frac{\psi^2(t)}{t^2}\,dt \int_{-\infty}^{\infty} \left\{l\left(\frac{t+\epsilon}{t-\epsilon}\right)\right\}^2 dt.$$

The second factor is $O(\epsilon)$, while the first is

$$\int_{2\epsilon}^{\infty} \frac{\Psi'(t)}{t^2}\,dt = -\frac{\Psi(2\epsilon)}{4\epsilon^2} + 2\int_{2\epsilon}^{\infty} \frac{\Psi(t)}{t^3}\,dt \leqslant 2\int_{2\epsilon}^{\infty} \frac{\Psi(t)}{t^3}\,dt$$

$$= 2\int_{2\epsilon}^{1} o\left(\frac{1}{t^2}\right) dt + 2\int_{1}^{\infty} O\left(\frac{1}{t^3}\right) dt = o\left(\frac{1}{\epsilon}\right).$$

Hence $M_2(\epsilon) \to 0$.†

## Proof of Theorem 2

**10.** Let

$$f_1 = \frac{1}{t-x} \quad (|t-x| \geqslant \epsilon), \qquad f_1 = 0 \quad (|t-x| < \epsilon).$$

The transform $g_1$ of $f_1$ is

$$\frac{1}{\pi}\left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty}\right) \frac{du}{(u-t)(u-x)} = \frac{1}{\pi(t-x)} l\left(\frac{t-x+\epsilon}{t-x-\epsilon}\right).$$

Hence, by (8.1),

$$\left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty}\right) \frac{f(t)}{t-x}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(t)}{t-x} l\left(\frac{t-x+\epsilon}{t-x-\epsilon}\right) dt.$$

Making $\epsilon \to 0$, and using Theorem 3, we find that

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt = \pi g(x)$$

for almost all $x$. This is (1.2.1), and (1.2.2) is proved similarly.

† In this note I am concerned only with functions of $L^2$; but it is worth remarking that Theorem 3 is true for functions of $L^p$ whenever $p > 1$, and that the proof generalizes immediately. The result is true at a point of continuity, but *not* necessarily almost always, when $p = 1$. See Hardy (**3**).

## An alternative method

**11.** The alternative proof of Theorem 1 suggested by Professor Titchmarsh,† and referred to in § 1, proceeds as follows.

Suppose first that $f(t)$ has a continuous derivative and vanishes for $|t| > T$; we may call such a function a function of class $\mathbf{F}_T$. Then

$$\phi(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,l\left(1-\frac{x}{t}\right)dt = -\frac{x}{\pi}\int_{-\infty}^{\infty} f(xu)\,l\left(1-\frac{1}{u}\right)du,$$

and so

$$g(x) = \phi'(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} f(xu)\,l\left(1-\frac{1}{u}\right)du - \frac{x}{\pi}\int_{-\infty}^{\infty} uf'(xu)\,l\left(1-\frac{1}{u}\right)du$$

$$= -\frac{1}{x\pi}\int_{-\infty}^{\infty} F(t)\,l\left(1-\frac{x}{t}\right)dt,$$

where

$$F(t) = f(t)+tf'(t).$$

The transformations are trivial when $f$ belongs to $\mathbf{F}_T$.

We now have

$$\int_{-\infty}^{\infty} g^2(x)\,dx = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{dx}{x^2}\int_{-\infty}^{\infty} F(t)\,l\left(1-\frac{x}{t}\right)dt\int_{-\infty}^{\infty} F(u)\,l\left(1-\frac{x}{u}\right)du$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty} F(t)\,dt\int_{-\infty}^{\infty} F(u)\,du\int_{-\infty}^{\infty} l\left(1-\frac{x}{t}\right)l\left(1-\frac{x}{u}\right)\frac{dx}{x^2} \quad (11.1)$$

(the triple integral being absolutely convergent for $f$ of $\mathbf{F}_T$). If we write $1/x$ for $x$, the inner integral becomes a special case of the integral of Lemma 2, and its value is

$$\pi^2\min\left(\frac{1}{|t|},\frac{1}{|u|}\right)$$

if $t$ and $u$ have the same sign, 0 if $t$ and $u$ have opposite signs.

Suppose that $0 < u < t$. Then the corresponding part of (11.1) is

$$\int_0^{\infty} F(t)\,dt\int_0^{t}\frac{F(u)}{t}\,du = \int_0^{\infty}\frac{f(t)+tf'(t)}{t}\,dt\int_0^{t}\{f(u)+uf'(u)\}\,du$$

$$= \int_0^{\infty}(f^2+tff')\,dt = \tfrac{1}{2}\int_0^{\infty} f^2\,dt,$$

† In a rather different setting suggested by Watson's work.

by partial integration. The part for which $0 < t < u$ gives the same contribution, and the parts for which $t$ and $u$ are both negative contribute corresponding integrals over $(-\infty, 0)$. Hence

$$\int_{-\infty}^{\infty} g^2(x)\,dx = \int_{-\infty}^{\infty} f^2(t)\,dt. \tag{11.2}$$

This is (2.1) for $f$ of $\mathbf{F}_T$. But any $f$ of $L^2$ is a mean-square limit of a sequence of functions $f_n$ of $\mathbf{F}_T$, and (11.2) may therefore be extended to all $f$ of $L^2$ in the ordinary manner. The formulae (1.1) may then be deduced from (2.1) as (1.1.2) was deduced in § 8.

[Mr. J. Cossar suggests to me another interesting alternative to the argument of § 5. The function

$$F(t,u) = F(t-u) = \int_{-\infty}^{\infty} f(t+x)f(u+x)\,dx$$

is continuous. If we write $\Delta(x,h)$ in the form

$$\Delta(x,h) = \frac{1}{2\pi h}\int_{-\infty}^{\infty} f(t+x)\,l\left(\frac{t+h}{t-h}\right)dt,$$

and evaluate $I(h,h')$ with the help of Lemma 2, we find

$$I(h,h') = \frac{1}{8h\bar{h}'}\int_{-\infty}^{\infty} F(t)\{|t+h+h'|+|t-h-h'|-|t+h-h'|-|t-h+h'|\}\,dt;$$

and, $F(t)$ being continuous, it is easy to show that $I(h,h') \to F(0)$.]

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# A NEW METHOD OF FINDING THE POTENTIALS OF UNIFORM SOLID ELLIPSOIDS AND ELLIPTIC CYLINDERS

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THE method consists in applying Green's formula,

$$\iiint (U\nabla^2 V - V\nabla^2 U)\, d\tau = \iint \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) dS,$$

where
$$U \equiv 1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u},$$
$$V = \frac{1}{r} = [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-\frac{1}{2}},$$

$\xi$, $\eta$, $\zeta$ being the coordinates of the attracted point; and the integrations are taken through the volume and over the surface of the ellipsoid $U = 0$.

It is convenient to begin by proving certain lemmas.

I. Let $T$ be the volume of the ellipsoid $U = 0$; then

$$T = \tfrac{4}{3}\pi\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}},$$

and
$$\frac{2}{T}\frac{dT}{du} = \sum \frac{1}{a^2+u} = -\tfrac{1}{2}\nabla^2 U, \text{ a constant.}$$

II. Let $l'$, $m'$, $n'$ be the direction-cosines of the outward-drawn normal at any point of the ellipsoid, and let $dn$ denote differentiation along the normal. Then

$$l' = \frac{px}{a^2+u}, \qquad m' = \frac{py}{b^2+u}, \qquad n' = \frac{pz}{c^2+u},$$

where
$$\frac{1}{p^2} = \sum \frac{x^2}{(a^2+u)^2},$$

and
$$\frac{\partial U}{\partial n} = \sum l' \frac{\partial U}{\partial x} = -2\sum \frac{l'x}{a^2+u} = -\frac{2}{p}.$$

Again
$$U \equiv 1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u};$$

therefore
$$\delta U = -\sum \frac{2x\delta x}{a^2+u} + \sum \frac{x^2}{(a^2+u)^2}\,\delta u.$$

If $\delta t$ is the normal thickness at $(x, y, z)$ of the shell bounded by $U = 0$, $U+\delta U = 0$,

$$\delta x = l'\delta t, \qquad \delta y = m'\delta t, \qquad \delta z = n'\delta t, \qquad \delta U = 0,$$

and

$$\frac{1}{p^2}\delta u = \frac{2}{p}\,\delta t.$$

This gives* $$\frac{2}{p} = 4\frac{\delta t}{\delta u}, \quad \text{or} \quad \frac{\partial U}{\partial n} = -4\frac{dt}{du}.$$

## Maclaurin's theorem

Let $P(\xi, \eta, \zeta)$ be an external point:

$\nabla^2(1/r) = 0$ throughout the volume, $U = 0$ on the surface,

and we get $$\frac{4}{T}\frac{dT}{du}\iiint\frac{d\tau}{r} = 4\iint\frac{1}{r}\frac{dt}{du}\,dS.$$

But $dtdS$ is an element of the volume of the shell, $(1/r)dtdS$ is its potential at $P$ and $du$ is constant over $S$; therefore $\int dtdS/r$ is the potential of the shell and

$$\iint\frac{1}{r}\frac{dt}{du}\,dS = \frac{d}{du}\iiint\frac{d\tau}{r}.$$

Thus, if $\Omega$ is the potential of the ellipsoid, supposed of unit density,

$$\frac{\Omega}{T}\frac{dT}{du} = \frac{d\Omega}{du},$$

so that $\Omega/T$ is constant.

This is Maclaurin's theorem that the potentials of confocal ellipsoids (or confocal shells) at any point external to them are proportional to their volumes (or masses).

## Internal points

Let $P$ be an internal point. We take $P$ in a small cavity and add to the right-hand side

$$-\iint\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)dS',$$

taken over the surface of the cavity.

The volume and exterior surface integrals are known to be con-

* The result may also be obtained by differentiating

$$p^2 = a^2l'^2+b^2m'^2+c^2n'^2+u.$$

vergent and may be extended up to the point $P$;

$$-\iint U\frac{\partial V}{\partial n}\,dS' = \iint \frac{U\cos\theta}{r^2}\,dS' \text{ (say)} = \iint U\,d\omega,$$

and this tends to $4\pi U(\xi,\eta,\zeta)$ as the cavity is indefinitely diminished;

$$\iint V\frac{\partial U}{\partial n}\,dS' = \iint \frac{\partial U}{\partial n}\,r\sec\theta\,d\omega,$$

which tends to zero with $r$. Green's equation thus becomes:

$$\frac{d\Omega}{du}-\frac{\Omega}{T}\frac{dT}{du}+\pi U = 0,$$

whence

$$\frac{\Omega}{T}+\pi\int^{u}\frac{U}{T}\,du = \text{constant},$$

or

$$\left(\frac{\Omega}{T}\right)_u-\left(\frac{\Omega}{T}\right)_\infty = \pi\int_u^\infty\frac{U}{T}\,du.$$

The potential of the ellipsoid lies between the potentials of spheres of radii $\sqrt{(a^2+u)}$ and $\sqrt{(c^2+u)}$, and these, if $P$ is at a distance $r$ from the centre, are

$$2\pi(a^2+u-r^2/3), \qquad 2\pi(c^2+u-r^2/3),$$

and the ratios of these to $\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}$ tend to zero as $u$ increases indefinitely. Hence

$$\Omega/T\to 0 \text{ as } u\to\infty.$$

Thus

$$\frac{\Omega}{T} = \pi\int_u^\infty\frac{U}{T}\,du,$$

i.e.

$$\Omega/\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}$$

$$= \pi\int_u^\infty\left(1-\frac{\xi^2}{a^2+u}-\frac{\eta^2}{b^2+u}-\frac{\zeta^2}{c^2+u}\right)\frac{du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}.$$

If the axes of the ellipsoid are $2a$, $2b$, $2c$, we get

$$\Omega = \pi abc\int_0^\infty\left(1-\sum\frac{\xi^2}{a^2+u}\right)du\,\{\prod(a^2+u)\}^{-\frac{1}{2}}$$

for the potential at an internal point.

### External points

To find the potential at an external point, we follow the usual method of making the potential at that point depend on the potential at an internal point. We draw through $P$ the ellipsoid confocal with $U=0$, and, since the potential is continuous at all points, we may

regard $P$ as an internal point of this confocal and obtain the relation

$$\frac{\Omega_\lambda}{\pi\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}} = \int_\lambda^\infty \left(1 - \sum \frac{\xi^2}{a^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}$$

where $\lambda$ is the positive root of the equation $\sum \frac{\xi^2}{a^2+u} = 1$. But by Maclaurin's theorem, $\Omega_0/\pi abc = \Omega_u/\pi a'b'c'$ say, so that

$$\Omega = \pi abc \int_\lambda^\infty \left(1 - \sum \frac{\xi^2}{a^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)(c^2+u)\}^{\frac{1}{2}}}$$

gives the potential of the ellipsoid bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ at an external point.

## Application to an elliptic cylinder

The attraction of a uniform thin rod of mass $m$ per unit length being $2m/r$, the proper form for $V$ in Green's equation is $2\log(r_0/r)$, where $r_0$ is some constant; $U$ is now

$$1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u}.$$

The lemmas and propositions already proved hold *mutatis mutandis*, and we again arrive at Maclaurin's theorem that the potentials at an external point of confocal cylinders (of unit density) are as the areas of their cross-sections.

If the point is internal, it must be surrounded by a small circular cavity; the integrals are continuous as before and we get the same results, since $-\int U \frac{\partial V}{\partial n} ds'$, where $ds'$ is an element of arc of the small circle, is $\int U \frac{2}{r} r\, d\theta$, which is $4\pi U(\xi, \eta, \zeta)$.

Thus the same equation holds as before, viz.:

$$\frac{d}{du} \frac{\Omega}{\pi\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}} + \left(1 - \frac{\xi^2}{a^2+u} - \frac{\eta^2}{b^2+u}\right)\{(a^2+u)(b^2+u)\}^{-\frac{1}{2}} = 0.$$

On integration

$$\frac{\Omega_u}{\pi\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}} - \frac{\Omega_\gamma}{\pi\{(a^2+\gamma)(b^2+\gamma)\}^{\frac{1}{2}}} = \int_u^\gamma \left(1 - \sum \frac{\xi^2}{a^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}.$$

It is easy to prove by means of Laplace's and Poisson's equations or by direct integration that the potentials of a circular cylinder of radius $a$ are:

$$\pi a^2 \log \frac{r_0^2}{r^2} \qquad \text{at an external point,}$$

$$\pi a^2 \log \frac{r_0^2}{a^2} + \pi(a^2 - r^2) \qquad \text{at an internal point,}$$

since

$$\int_0^\pi \log(a^2 - 2ab\cos\theta + b^2)\, d\theta = \pi \log a^2 \qquad (a > b);$$

and that the potential of an elliptic cylinder at a point on its axis is

$$\pi ab \log \frac{r_0^2}{\{\frac{1}{2}(a+b)\}^2} + \pi ab,$$

since

$$\int_0^\pi \log(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta = \pi \log\left(\frac{a+b}{2}\right)^2.$$

Thus for the potential of a circular cylinder at an internal point

$$\Omega/\pi a^2 + \log \frac{a^2}{r_0^2} = 1 - \frac{r^2}{a^2},$$

and tends to 1 as $a$ tends to $\infty$. Also, for the potential of an elliptic cylinder at a point on its axis

$$\Omega/\pi ab + \log\left\{\frac{\frac{1}{2}(a+b)}{r_0}\right\}^2 = 1.$$

On noting that the potential of an elliptic cylinder lies in value between the potentials of circular cylinders of radii $a$ and $b$, it is not difficult to prove that as $a^2$ and $b^2$ tend to infinity, whilst differing by a finite quantity, the property,

$$\Omega/\pi ab + \log\left\{\frac{\frac{1}{2}(a+b)}{r_0}\right\}^2 \to 1,$$

is true of the potential at any internal point of the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus in general; as $\gamma \to \infty$,

$$\frac{\Omega_u}{\pi(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}} + \log\left\{\frac{(a^2+\gamma)^{\frac{1}{2}}+(b^2+\gamma)^{\frac{1}{2}}}{2r_0}\right\}^2$$

$$= 1 + \int_u^\gamma \left(1 - \frac{\xi^2}{a^2+u} - \frac{\eta^2}{b^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}.$$

This may be extended to the potential at an external point by Maclaurin's theorem, and we find:

I. The potential of the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ at an internal point is

$$\pi ab \lim_{\gamma\to\infty}\left[1 - \log\left\{\frac{(a^2+\gamma)^{\frac{1}{2}}+(b^2+\gamma)^{\frac{1}{2}}}{2r_0}\right\}^2 + \right.$$

$$\left. + \int_0^\gamma \left(1 - \frac{\xi^2}{a^2+u} - \frac{\eta^2}{b^2+u}\right) \frac{du}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}\right].$$

II. The potential at an external point is obtained from the above by writing $\lambda$ instead of zero as the lower limit of the integral, where $\lambda$ is the positive root of the equation

$$\frac{\xi^2}{a^2+u} + \frac{\eta^2}{b^2+u} = 1.$$

The integrals in these expressions are easily evaluated, and we find that

$$\frac{\Omega_u}{\pi a'b'} + \log\left(\frac{a'+b'}{2r_0}\right)^2 \equiv 1 - \frac{2\xi^2}{a'(a'+b')} - \frac{2\eta^2}{b'(a'+b')},$$

where $a'^2 = a^2+u$, $b'^2 = b^2+u$.

It follows that the components of attraction of the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are

$$-\frac{4\pi ab}{a'+b'}\frac{\xi}{a'}, \qquad -\frac{4\pi ab}{a'+b'}\frac{\eta}{b'},$$

where $a'$ and $b'$ are respectively $a$ and $b$ for an internal point, $\sqrt{(a^2+\lambda)}$, $\sqrt{(b^2+\lambda)}$ for an external point.

These results suggest another line of proof which avoids the rather tedious evaluation of

$$\lim_{\gamma\to\infty}\left[\frac{\Omega_\gamma}{\pi(a^2+\gamma)(b^2+\gamma)^{\frac{1}{2}}} + \log\left\{\frac{(a^2+\gamma)^{\frac{1}{2}}+(b+\gamma)^{\frac{1}{2}}}{2r_0}\right\}^2\right].$$

We obtain the expressions for the attraction at an internal point by direct evaluation. Then, by using the value already given for the potential on the axis, we see that the potential at an internal point of the cylinder $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}=1$ is given by the equation

$$\frac{\Omega_u}{\pi(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}}+\log\left\{\frac{(a^2+u)^{\frac{1}{2}}+(b^2+u)^{\frac{1}{2}}}{2r_0}\right\}^2$$

$$=1-\int_u^\infty\left(\frac{\xi^2}{a^2+u}+\frac{\eta^2}{b^2+u}\right)\frac{du}{\{(a^2+u)(b^2+u)\}^{\frac{1}{2}}}.$$

This can be put into the form previously obtained, and the result can then be extended by means of Maclaurin's theorem to the potential at an external point.

# DERIVATE PLANES AND TANGENT PLANES OF A MEASURABLE FUNCTION

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**1.** A VERY full investigation of the possible values of the four derivates of a function of one variable has been made* by Denjoy and by Mrs. Young. Their main results can be summarized briefly as follows. The upper right-hand derivate of a function $f(x)$ is defined as

$$D^+f(x) = \lim_{\rho \to 0} \overline{\text{bound}}_{0<h<\rho} \frac{f(x+h)-f(x)}{h};$$

and there are corresponding definitions for the other three derivates, $D_+$, $D^-$, and $D_-$. Then it is proved that, if the function $f(x)$ is measurable, almost all points of $y = f(x)$ are of one of three types. Either (i) there is a unique tangent at the point (that is, all four derivates have the same finite value); or (ii) the upper derivate on one side and the lower derivate on the other side have the same finite value, whilst one of the remaining pair is $+\infty$ and the other is $-\infty$; or (iii) both the upper derivates are $+\infty$ and both the lower derivates are $-\infty$.

It is the object of this paper to establish corresponding results for functions of two variables. The question of the existence of derivate planes for such functions does not appear to have been discussed previously, though Rademacher and Stepanoff † have obtained the criterion for the existence of a tangent plane. They write

$$L_f(x,y) = \lim_{\rho \to 0} \overline{\text{bound}}_{(h^2+k^2)<\rho} \left| \frac{f(x+h,y+k)-f(x,y)}{\sqrt{(h^2+k^2)}} \right|,$$

and prove that *if $L_f(x,y)$ is finite in a set of points $E$, then $f(x,y)$ is totally differentiable in the sense of Stolz‡ almost everywhere in $E$.* This theorem is a corollary of Theorem 4*b*.

* A. Denjoy, *Jour. de Math.* (7th series), 1 (1915), 105–240; G. C. Young, *Proc. London Math. Soc.* (2), 15 (1916), 360–84.

† H. Rademacher, *Math. Ann.* 79 (1919), 340–59; W. Stepanoff, *Math. Ann.* 90 (1923), 318–20; also *Recueil de la Soc. Math. de Moscou*, 32 (1924), 511–26. Further extensions to restricted classes of functions are given by D. Menchoff, *Math. Ann.* 105 (1931), 75–85.

‡ According to Stolz's definition, a function $f(x, y)$ is 'totally differentiable' at a point $(x_0, y_0)$ if there exist numbers $A$ and $B$ such that

The theorems proved below arise out of a collaboration with Mr. Burkill in discussing various problems of differentiation. We have found that there are greater difficulties than one might expect in obtaining theorems of the Denjoy-Young type which apply to the whole class of measurable functions of two variables. If we adopt an obvious method of approach to the theory of derivate planes and define 'partial derivates', as in § 14 below, we can deal only with functions of a rather restricted class. This restriction, which has been overlooked by some previous writers,* is due to considerations of measurability. Although, as Neubauer† has proved, partial derivates are measurable if the original function belongs to a Baire class, yet an example constructed by Hahn shows that there exist other measurable functions with non-measurable partial derivates. We are compelled, therefore, to choose a different starting-point. The fundamental idea of this paper, which enables us to include the whole field of measurable functions, is that of the 'directed' derivates defined in the next section. These prove to be measurable and play a part similar to that played by the four derivates in the Denjoy-Young theorems.

**2.** Let $z = x+iy$, and let‡ $f(x, y) = f(z)$ be a measurable function of $x$ and $y$. Let $S_\alpha(\rho, \eta)$ denote the open region consisting of points $z = re^{i\theta}$ for which $0 < r < \rho$, $\alpha-\eta < \theta < \alpha+\eta$. We write

$$B^\alpha f = B^\alpha(\rho, \eta, z) = \overline{\text{bound}}_{h \in S_\alpha(\rho,\eta)} \frac{f(z+h)-f(z)}{|h|\cos\phi}, \tag{2.1}$$

where $h = k+il = |h|e^{i(\alpha+\phi)}$.

It is clear that $B^\alpha$ is an increasing function of $\rho$ and $\eta$; hence the following limits exist:

$$\Delta^\alpha(\eta, z) = \lim_{\rho \to 0} B^\alpha(\rho, \eta, z),$$
$$D^\alpha(z) = \lim_{\eta \to 0} \Delta^\alpha(\eta, z).$$

$$f(x_0+r\cos\theta, y_0+r\sin\theta)-f(x_0, y_0)-Ar\cos\theta - Br\sin\theta = o(r)$$

uniformly in $\theta$ as $r \to 0$. We can therefore say that there is a tangent plane to $z = f(x, y)$ at such a point. $A$ and $B$ are, of course, the partial derivatives of $f(x, y)$.

* See, for example, G. C. Young, *Proc. London Math. Soc.* (2), 20 (1922), 182–8. The theorems proved here are not true for all measurable functions.

† M. Neubauer, *Monatshefte für Math. und Phys.* 38 (1931), 139–46. Hahn's example is quoted in this paper.

‡ The double notation (real and complex) is introduced for the sake of brevity. It is to be remembered that, whichever notation is used, $f$ is a real one-valued measurable function.

The function $D^\alpha(z)$ may be regarded as a 'directed' upper derivate. A similar process, with the substitution of the lower bound for the upper in (2.1), defines $B_\alpha$, $\Delta_\alpha$, and, finally, the directed lower derivate, $D_\alpha$.

[The factor $\cos\phi$ which appears in the denominator in (2.1) may be omitted without affecting the value of $D^\alpha$: its inclusion simplifies both the proof of measurability and the proofs of later theorems.]

The consideration of the relations between directed derivates yields considerable information as to the structure of a measurable function of two variables. We first, in §§ 3 and 4, discuss the nature of the directed upper derivate as a function of $z$ and of $\alpha$: it is shown that, for a fixed $\alpha$, $D^\alpha(z)$ is a measurable function of $z$, and that, for fixed $z$, it is an upper semi-continuous function of $\alpha$. Theorems 1, 2, and 3 describe the symmetry which exists between two 'opposite' derivates, $D^\alpha(z)$ and $D^{\alpha+\pi}(z)$. Theorem 4 and its corollaries in §§ 10–12 extend this symmetry to the whole set of derivates at a point: it is proved that, if exception is made of a set of values of $z$ of measure zero, either $D^\alpha(z) = +\infty$ for all values of $\alpha$, or $D^\alpha(z) = \cos\alpha D^0(z) + \sin\alpha D^{\frac{1}{2}\pi}(z)$. A complete summary of the information obtained is given in § 13, and it is shown that the Rademacher-Stepanoff theorem and other results are contained in our theorems.

**3. The measurability of $D^\alpha(z)$.** We first prove that, if $\alpha$ is any fixed number, the set in which $B^\alpha(z) > K$ is measurable. Write $F(z) = f(z) - K(x\cos\alpha + y\sin\alpha)$. The set in which $B^\alpha f > K$ is identical with the set in which $B^\alpha F > 0$; that is, with the set $e$ in which

$$\Phi(z) = \overline{\underset{h\epsilon S_\alpha(\rho,}{\text{bound}}}\ F(z+h) > F(z). \tag{3.1}$$

It is easily seen that, when $z' \to z$, $\liminf \Phi(z') \geqslant \Phi(z)$; that is, $\Phi(z)$ is a lower semi-continuous function of $z$ and is therefore measurable. Since $F(z)$ is also measurable, the set $e$ in which (3.1) holds is measurable. The set $e$ is the set in which $B^\alpha f > K$; it follows that $B^\alpha$ is measurable.

In the definition of $\Delta^\alpha$ we may replace the continuous parameter $\rho$ by any sequence of numbers tending to zero. Thus $\Delta^\alpha$ is the measurable limit function of a sequence of measurable functions. A similar argument shows that $D^\alpha$ is measurable.

The measurability of $D_\alpha(z)$ can be established in the same manner.

**4. $D^\alpha$ as a function of $\alpha$.** We make use later of the fact that, if $z_0$ is a fixed point and $K$ is any given number, the values of $\alpha$ for which $D^\alpha(z_0) \geqslant K$ form a closed set; in other words, $D^\alpha$ is an upper semi-continuous function of $\alpha$. To prove this, let $\alpha$ be the limit of a sequence of numbers $\alpha_1, \alpha_2, \ldots$, for which $D^{\alpha_n} \geqslant K$. Given any positive numbers $\epsilon, \rho, \eta_n$, we can find in $S_{\alpha_n}(\rho, \eta_n)$ at least one point $h_n$ such that

$$f(z_0+h_n)-f(z_0) > (K-\epsilon)|h_n|\cos\eta_n. \tag{4.1}$$

Every sector $S_\alpha(\rho, \eta)$ contains an infinity of lines $\operatorname{am} z = \alpha_n$. Choose any one, say $\alpha_r$, and let $\eta_r$ be so small that $S_{\alpha_r}(\rho, \eta_r)$ lies entirely inside $S_\alpha(\rho, \eta)$; we may suppose also that $\cos\eta_r$ is greater than $(1-\epsilon)$. The point $h_r$, chosen as in (4.1), therefore lies inside $S_\alpha(\rho, \eta)$, and so

$$B^\alpha(\rho, \eta, z_0) > (K-\epsilon)(1-\epsilon).$$

This is true for every positive $\rho$, $\eta$, and $\epsilon$. Therefore $D^\alpha \geqslant K$.

It follows easily that we can replace $K$ by any continuous function $\phi(\alpha)$ and show that the values of $\alpha$ for which $D^\alpha \geqslant \phi(\alpha)$ form a closed set. We use these results in the form of the following lemma:

LEMMA A. *Suppose that $\phi(\alpha)$ is a continuous function of $\alpha$ and that $D^\alpha(z_0) \geqslant \phi(\alpha)$ for a set of values of $\alpha$ everywhere dense in the interval $(-\pi, \pi)$. Then the inequality holds for all values of $\alpha$.*

**5.** THEOREM 1. *For any given values of $\alpha$ and $K$, if $D^\alpha(z) < K$ in a set $E$, then $D^{\alpha+\pi}(z) > -K$ almost everywhere in $E$.*

Without loss of generality we can take $\alpha$ to be zero. If possible let there be a set $e$, of measure greater than zero, in which $D^0 < K$, $D^\pi \leqslant -K$. We can choose $\delta, \eta, \rho$ successively so small that the sets in which

$$\begin{aligned} &\{D^0 < K-5\delta, \quad D^\pi \leqslant -K\}, && (e_1)\\ &\{\Delta^0(\eta) < K-4\delta, \quad \Delta^\pi(\eta) < -K+\delta\}, && (e_2)\\ &\{B^0(\rho, \eta) < K-3\delta, \quad B^\pi(\rho, \eta) < -K+2\delta\}, && (e_3) \end{aligned} \tag{5.1}$$

are all of measure greater than zero. Let $z_0 = x_0+iy_0$ be a point of density* of $e_3$. We can then find a point $z_1 = x_1+iy_1$ of $e_3$ such that $(z_1-z_0)$ belongs to $S_0(\rho, \eta)$. Then, by (5.1),

$$\frac{f(z_1)-f(z_0)}{x_1-x_0} < K-3\delta.$$

* i.e. a point of metric density 1. See, for example, E. W. Hobson, *Theory of Functions of a Real Variable*, 3rd ed. (Cambridge, 1921), vol. i, 194, where it is shown that almost all points of a measurable set are points of density of the set in this sense.

Also $(z_0-z_1)$ belongs to $S_\pi(\rho, \eta)$; therefore

$$\frac{f(z_0)-f(z_1)}{x_1-x_0} < -K+2\delta.$$

The truth of the theorem follows from the inconsistency of these two inequalities.

COROLLARY. *If $D^\alpha$ is finite in a set $E$, then $D^\alpha+D^{\alpha+\pi} \geqslant 0$ almost everywhere in $E$.*

The proof is easily constructed. The further corollary that 'if $D^\alpha = -\infty$ in $E_1$, then $D^{\alpha+\pi} = +\infty$ almost everywhere in $E_1$', is trivial. We can in fact show (Theorem 2, Corollary) that the set $E_1$ is of measure zero in any case.

**6.** The proofs of the next two theorems are essentially similar to those given by Mrs. Young in the discussion of the corresponding one-variable theorems. Sundry additional complications make it necessary to reproduce the whole argument. Both proofs depend on a lemma* of the 'covering' type which we quote without proof.

LEMMA Y. *If each point of an interval is the left-hand end-point of a finite or infinite set of intervals, we can find a finite number of these intervals, nowhere overlapping, such that the sum of the complementary intervals is less than any pre-assigned quantity.*

**7.** THEOREM 2. *If $D^\alpha(z) = +\infty$ in a set $E$, then $D^{\alpha+\pi}(z) = +\infty$ almost everywhere in $E$.*

It is sufficient, as in Theorem 1, to establish the result when $\alpha = 0$. Suppose, if possible, that there is a set of positive measure in which $D^0 = +\infty$ and $D^\pi < +\infty$. Let $K$ be chosen so large that the set in which $D^0 = +\infty$, $D^\pi < \frac{1}{2}K$ is of positive measure. Finally, choose $\rho$ and $\eta$ so small that

$$D^0(z) = +\infty, \qquad B^\pi(\rho, \eta, z) < K \tag{7.1}$$

in a set $E_1$ of positive measure.

Let $E_1(y')$ be the linear set consisting of the points of $E_1$ which lie on the line $y = y'$. By Fubini's theorem on plane measure there is at least one value of $y'$, say $\beta$, such that $E_1(\beta)$ is of linear measure greater than zero; for the sake of brevity we shall write $x$ instead of $(x+i\beta)$ where this is possible without ambiguity. We can find a perfect set $P$, of positive measure, contained in $E_1(\beta)$: let $P_1(a, b)$ denote the

* W. H. and G. C. Young, *Proc. London Math. Soc.* (2), 9 (1911), 325–35 (327).

portion of $P$ contained in the closed interval $a \leqslant x \leqslant b$. We can choose $a$ and $b$ so that

(i) $a$ and $b$ belong to $P_1 = P_1(a,b)$,

(ii) $(1+\eta)(b-a) < \rho$,

(iii) meas $P_1 > (1-\delta)(b-a)$.

We proceed to associate with each point $x$ of the interval $(a,b)$ a set of intervals of which $x$ is the left-hand end-point. For this purpose we distinguish between those points which are points of $P_1$ and those which are not.

(i) Let $x+i\beta$ belong to $P_1$. Let $Q$ be a large positive number; it will be sufficient if

$$Q > \frac{1}{1-2\delta}\left\{(1+\eta)\frac{f(b,\beta)-f(a,\beta)}{b-a} + K(\eta+2\delta)\right\}. \tag{7.2}$$

Since $D^0(x+i\beta) = +\infty$ there are an infinity of numbers $k$ and $l$ such that $|l| < \eta^2 k$ and that

$$\frac{f(x+k,\beta+l)-f(x,\beta)}{k} > Q.$$

With such points $x+i\beta$ we associate all the intervals $\{x, x+(1+\eta)k\}$ on the line $y = \beta$. Let $\Delta$ denote the complete set of such intervals for all points of $P_1$.

(ii) Let $x+i\beta$ not belong to $P_1$; then it is a point of one of the complementary intervals of the perfect set, say of $(\xi,\xi')$. With such points we associate the right-hand portion of the complementary interval in which they lie; that is, $(x,\xi')$. Let $\Delta'$ denote the complete set of those intervals.

The combined sets $\Delta$ and $\Delta'$ satisfy the conditions of Lemma Y; we can therefore select from them a finite non-overlapping set of intervals to cover a total length greater than $(1-\delta)(b-a)$. This finite set can be divided into two parts, $\Delta_1$ and $\Delta_1'$, consisting of intervals of $\Delta$ and $\Delta'$ respectively. Since the intervals of $\Delta_1'$ contain no points of $P_1$, we have

$$\text{meas}\,\Delta_1' \leqslant (b-a) - \text{meas}\,P_1 < \delta(b-a);$$

$$\text{meas}\,\Delta_1 > (1-\delta)(b-a) - \text{meas}\,\Delta_1' > (1-2\delta)(b-a).$$

Finally we consider the set of intervals $\Delta_2$ obtained by reducing each interval of $\Delta_1$ from $\{x, x+(1+\eta)k\}$ to $(x, x+k)$. Then

$$\text{meas}\,\Delta_2 > \frac{1-2\delta}{1+\eta}(b-a), \tag{7.3}$$

and no two intervals of $\Delta_2$ are contiguous. Furthermore we can suppose that one interval of $\Delta_2$ has its left-hand end-point at $a$: if there is not already such an interval one can be added since $a$ belongs to $P_1$.

Let the intervals of $\Delta_2$ be $(a, a_1)$, $(a_2, a_3)$,..., $(a_{2n-2}, a_{2n-1})$. Then $a$, $a_2$,..., $a_{2n}$ $(= b)$ are points of $P_1$, and, by considering the method of forming $\Delta_2$ from $\Delta_1$, we see that $(a_{2r}-a_{2r-1}) \geqslant \eta(a_{2r-1}-a_{2r-2})$. Since $(a_{2r-2}, a_{2r-1})$ belongs to $\Delta_2$ there is a number $l_r$ such that

$$f(a_{2r-1}, \beta+l_r)-f(a_{2r-2}, \beta) > Q(a_{2r-1}-a_{2r-2}), \tag{7.4}$$

$$|l_r| < \eta^2(a_{2r-1}-a_{2r-2}) \leqslant \eta(a_{2r}-a_{2r-1}). \tag{7.5}$$

Considering next the interval $(a_{2r-1}, a_{2r})$ we see that, by virtue of (7.5), the point $(a_{2r-1}-a_{2r})+il_r$ belongs to $S_n(\rho, \eta)$. Therefore, by (7.1),

$$f(a_{2r-1}, \beta+l_r)-f(a_{2r}, \beta) < K(a_{2r}-a_{2r-1}). \tag{7.6}$$

Subtracting (7.6) from (7.4), we have

$$f(a_{2r}, \beta)-f(a_{2r-2}, \beta) > Q(a_{2r-1}-a_{2r-2})-K(a_{2r}-a_{2r-1}).$$

Summation from 1 to $n$ gives the inequality

$$\begin{aligned} f(b, \beta)-f(a, \beta) &> Q \operatorname{meas} \Delta_2 - K\{(b-a)-\operatorname{meas} \Delta_2\} \\ &> \left\{(Q+K)\frac{1-2\delta}{1+\eta}-K\right\}(b-a). \end{aligned} \tag{7.7}$$

Since this inequality is incompatible with (7.2) we conclude that there is no set of positive measure in which $D^0 = +\infty$ and $D^\pi < +\infty$.

COROLLARY. *For any assigned* $\alpha$, *the set of points at which* $D^\alpha(z) = -\infty$ *is of measure zero.*

This is an immediate result of combining Theorems 1 and 2.

**8.** THEOREM 3. *If, for a given value of* $\alpha$, $D^\alpha(z) < +\infty$ *in a set* $E$, *then* $D^\alpha(z)+D^{\alpha+\pi}(z) \leqslant 0$ *almost everywhere in* $E$.

Again we may take $\alpha$ to be zero. Since $D^0 < +\infty$ in $E$, it follows from Theorem 2 that $D^\pi < +\infty$ almost everywhere in $E$. Suppose, if possible, that $D^0+D^\pi > 0$ in a sub-set of $E$ of measure greater than zero. Choose $K$ so large and $k$, $\rho$, $\eta$ so small that

$$\begin{gathered} D^0+D^\pi > 2k, \\ B^0(\rho, \eta, z) < K, \quad B^\pi(\rho, \eta, z) < K, \quad \eta < k/8K, \\ D^0 > -K, \quad D^\pi > -K, \end{gathered} \tag{8.1}$$

in a set $E_1$ of measure greater than zero.

Divide the range $(-K, K)$ into $2m$ parts, $(-K, K_{-m+1})$,..., $(K_{-1}, 0)$,

$(0, K_1),\ldots,(K_{m-1}, K)$, such that each part is less than $k$. Since $m$ is finite we can choose $i$ so that

$$K_i < D^0 \leqslant K_{i+1}, \qquad D^\pi > 2k - K_{i+1}, \tag{8.2}$$

in a set $E_2$ of measure greater than zero.

We now argue as in Theorem 2. Let $\delta$ be a positive number (which we shall suppose to be less than $\eta$). We can find a line $y = \beta$ and a finite set of intervals $\Delta_2$ lying in an interval $(a, b)$ on this line with the following properties:

(i) The intervals of $\Delta_2$ being $(a, a_1)$, $(a_2, a_3),\ldots,(a_{2n-2}, a_{2n-1})$,

$$(a_{2r} - a_{2r-1}) \geqslant \eta(a_{2r-1} - a_{2r-2}) \qquad (a_0 = a;\ a_{2n} = b).$$

(ii) $$\operatorname{meas}\Delta_2 = \sum_1^n (a_{2r-1} - a_{2r-2}) > \frac{1-2\delta}{1+\eta}(b-a).$$

(iii) There are numbers $l_r$ such that

$$f(a_{2r-1}, \beta + l_r) - f(a_{2r-2}, \beta) > K_i(a_{2r-1} - a_{2r-2}), \tag{8.3}$$

$$|l_r| < \eta^2(a_{2r-1} - a_{2r-2}).$$

(iv) $$f(a_{2r-1}, \beta + l_r) - f(a_{2r}, \beta) < K(a_{2r} - a_{2r-1}). \tag{8.4}$$

Subtracting (8.4) from (8.3) and summing from 1 to $n$, we have

$$f(b, \beta) - f(a, \beta) > K_i \operatorname{meas}\Delta_2 - K\{(b-a) - \operatorname{meas}\Delta_2\}. \tag{8.5}$$

If now we use the second inequality of (8.2) and argue similarly from $b$ to $a$, we obtain the inequality

$$f(a, \beta) - f(b, \beta) > (2k - K_{i+1})\operatorname{meas}\Delta_2 - K\{(b-a) - \operatorname{meas}\Delta_2\}. \tag{8.6}$$

The addition of (8.5) and (8.6) gives

$$0 > \{2k - (K_{i+1} - K_i)\}\operatorname{meas}\Delta_2 - 2K\{(b-a) - \operatorname{meas}\Delta_2\}$$

$$> \{k(1-2\delta) - 2K(\eta + 2\delta)\}\frac{b-a}{1+\eta}.$$

Since $8K\eta < k$, $\delta < \eta$ and, obviously, $k < K$, this inequality is impossible. The contradiction establishes Theorem 3.

9. THEOREM 4. *Suppose that $D^0(z) < +\infty$, $D^{\frac{1}{2}\pi}(z) < +\infty$ in a set $E$. Then, almost everywhere in $E$,*

$$\lim_{\rho \to 0} \overline{\operatorname{bound}}_{|h|<\rho} \left\{\frac{f(z+h) - f(z) - kD^0(z) - lD^{\frac{1}{2}\pi}(z)}{|h|}\right\} \leqslant 0 \quad (h = k + il).$$

It follows from Theorem 3 and from the corollary to Theorem 2 that

$$D^\pi(z) = -D^0(z), \qquad D^{-\frac{1}{2}\pi}(z) = -D^{\frac{1}{2}\pi}(z), \tag{9.1}$$

and that these derivates are finite almost everywhere in $E$. Therefore, given any positive number $\delta$, we can choose $K$ so large that

$$|D_0(z)| < K, \qquad |D^{\frac{1}{2}\pi}(z)| < K,$$

and also (9.1) is true, in a set $E_1$ such that

$$\text{meas } E_1 > \text{meas } E - \delta.$$

Let $n$ be a positive integer and write $k = K/n$. Let $E_{r,s}$ be the sub-set of $E_1$ in which

$$(r-1)k < D^0 \leqslant rk, \qquad (s-1)k < D^{\frac{1}{2}\pi} \leqslant sk.$$

Given any positive number $\delta_1$ we can now choose $\rho_1$ and $\eta_1$ so small that, if $\rho \leqslant \rho_1$, $\eta \leqslant \eta_1$,

$$\begin{aligned} (r-1)k &< B^0(\rho, \eta) < (r+1)k, \\ (s-1)k &< B^{\frac{1}{2}\pi}(\rho, \eta) < (s+1)k, \end{aligned} \tag{9.2}$$

(with similar inequalities for $-B^{\pi}$ and $-B^{-\frac{1}{2}\pi}$) in a sub-set $e_{r,s}$ of $E_{r,s}$ such that

$$\text{meas } e_{r,s} > \text{meas } E_{r,s} - \frac{\delta_1}{4n^2}.$$

If $e'_{r,s}$ is the set of points of density of $e_{r,s}$, we have

$$\begin{aligned} \text{meas}\Big(\sum_{r,s} e'_{r,s}\Big) &= \text{meas}\Big(\sum_{r,s} e_{r,s}\Big) > \text{meas}\Big(\sum_{r,s} E_{r,s}\Big) - \delta_1 \\ &= \text{meas } E_1 - \delta_1 > \text{meas } E - \delta - \delta_1. \end{aligned} \tag{9.3}$$

Let $z_0 = x_0 + iy_0$ be a point of $e'_{r,s}$; $z_0$ is a point of density of $e_{r,s}$ so that we can choose $r_1 (< \rho_1)$ so small that, for a fixed value of $\eta$ $(\leqslant \eta_1)$,*

$$\text{meas}(e_{r,s};\ |z-z_0| < r_2) > (1-\sin^4\eta)\pi r_2^2 \qquad (r_2 < 2r_1). \tag{9.4}$$

(i) Let $z_1 = x_1 + iy_1$ be a point such that

$$|z_1 - z_0| < r_1, \qquad \eta \leqslant \text{am}(z_1 - z_0) \leqslant \tfrac{1}{2}\pi - \eta.$$

By (9.4) we can find a point of $e_{r,s}$ inside the circle with centre at $(x_1, y_0)$ and with radius $|z_1 - z_0|\sin^2\eta$: let $z' = x' + iy'$ be such a point. It is easily seen that $(z' - z_0)$ belongs to $S_0(\rho, \eta)$ and that $(z_1 - z')$ belongs to $S_{\frac{1}{2}\pi}(\rho, \eta)$. Hence, from (9.2),

$$\begin{aligned} f(z') - f(z_0) &< (r+1)k(x' - x_0), \\ f(z_1) - f(z') &< (s+1)k(y_1 - y'). \end{aligned} \tag{9.5}$$

Now

$$\begin{aligned} |x' - x_1| &< |z_1 - z_0|\sin\eta, \qquad |y' - y_0| < |z_1 - z_0|\sin\eta \\ |(r+1)k - D^0(z_0)| &< 2k, \qquad |(s+1)k - D^{\frac{1}{2}\pi}(z_0)| < 2k. \end{aligned}$$

* The set $(e_{r,s};\ |z-z_0| < r_2)$ is the set of points belonging to $e_{r,s}$ which lie inside the circle $|z-z_0| < r_2$.

Adding the two inequalities in (9.5), we have

$$\begin{aligned} f(z_1)-f(z_0) &< (r+1)k(x'-x_0)+(s+1)k(y_1-y') \\ &< D^0(z_0)(x'-x_0)+D^{\frac{1}{2}\pi}(z_0)(y_1-y')+4k|z_1-z_0| \\ &< D^0(z_0)(x_1-x_0)+D^{\frac{1}{2}\pi}(z_0)(y_1-y_0)+ \\ &\qquad +4k|z_1-z_0|+2K\sin\eta|z_1-z_0|. \end{aligned} \tag{9.6}$$

(ii) If $z_1$ is such that $|z_1-z_0| < r_1$, $0 \leqslant \mathrm{am}(z_1-z_0) < \eta$, then

$$\begin{aligned} f(z_1)-f(z_0) &\leqslant (x_1-x_0)B^0(r_1,\eta,z_0) \\ &< D^0(z_0)(x_1-x_0)+2k|z_1-z_0| \\ &< D^0(z_0)(x_1-x_0)+D^{\frac{1}{2}\pi}(z_0)(y_1-y_0)+ \\ &\qquad +K\sin\eta|z_1-z_0|+2k|z_1-z_0|. \end{aligned}$$

*A fortiori*, inequality (9.6) is true. Similar considerations show it to hold also in the remaining part of the first quadrant; namely, for $\frac{1}{2}\pi-\eta < \mathrm{am}(z_1-z_0) \leqslant \frac{1}{2}\pi$. Finally, the argument can be applied to the other three quadrants and, since (9.1) is true when $z = z_0$, the same inequality holds here also. Therefore, if $z_0$ is a point of $e'_{r,s}$,

$$\lim_{\rho\to 0}\ \overline{\text{bound}}_{|z-z_0|<\rho}\left\{\frac{f(z)-f(z_0)-D^0(z_0)(x-x_0)-D^{\frac{1}{2}\pi}(z_0)(y-y_0)}{|z-z_0|}\right\} < 2K\left(\sin\eta+\frac{2}{n}\right). \tag{9.7}$$

The inequality (9.7) holds for all relevant values of $r$ and $s$ and is therefore true whenever $z_0$ lies in the set $e' = \sum e'_{r,s}$. Considering that $\eta$, $\delta_1$, $1/n$ may be taken arbitrarily small, we see that, if $h = k+il$,

$$\lim_{\rho\to 0}\ \overline{\text{bound}}_{|h|<\rho}\left\{\frac{f(z_0+h)-f(z_0)-kD^0(z_0)-lD^{\frac{1}{2}\pi}(z_0)}{|h|}\right\} \leqslant 0, \tag{9.8}$$

when $z_0$ belongs to a set of points of measure greater than meas $E-\delta$. Finally, $\delta$ also is arbitrarily small: therefore (9.8) is true almost everywhere in $E$. Theorem 4 is thus proved.

**10.** For simplicity Theorem 4 has been enunciated and proved under the assumption that $D^0$ and $D^{\frac{1}{2}\pi}$ are finite. By a proof differing only in geometrical detail we could show that, if any two directed derivates $D^\alpha(z)$, $D^\beta(z)$ are finite in $E$ ($\alpha$ and $\beta$ being independent of $z$), then there is an inequality of the same type as (9.8) almost everywhere in $E$. In fact, it is sufficient to assume the finiteness of *one* derivate, say $D^\alpha(z)$. For, given any positive number $\delta$, we can then choose $K$, $\rho$, $\eta$ so that

$$B^\alpha(\rho,\eta,z) < K$$

in a set $E_1$ of measure greater than meas $E-\delta$. In this set $D^{\alpha+\frac{1}{2}\eta}$

and $D^{\alpha-\frac{1}{2}\eta}$ are finite: therefore we can establish an inequality of the type of (9.8) almost everywhere in $E_1$. Since $\delta$ is arbitrary the inequality exists almost everywhere in $E$. We have the following theorem:

THEOREM 4A. *For a given value of $\alpha$ suppose that $D^\alpha(z) < +\infty$ in a set $E$. We can find finite measurable functions $A(z)$, $B(z)$ such that*

$$\lim_{\rho\to 0} \overline{\text{bound}}_{|h|<\rho} \left\{\frac{f(z+h)-f(z)-kA(z)-lB(z)}{|h|}\right\} \leqslant 0 \qquad (h=k+il)$$

*almost everywhere in $E$.*

**11.** The result of Theorem 4A implies that, for the values of $z$ concerned,

$$D^\alpha(z) \leqslant \cos\alpha . A(z) + \sin\alpha . B(z), \tag{11.1}$$

for all values of $\alpha$. We can easily show that the equality sign actually holds for all values of $\alpha$, almost everywhere in $E$.

Choose any countable set of $\alpha$'s, $\alpha_1$, $\alpha_2$,..., everywhere dense in $(-\pi, \pi)$. Let the sub-set of $E$ in which

$$D^{\alpha_r}(z) < \cos\alpha_r . A(z) + \sin\alpha_r . B(z)$$

be $e_r$; by Theorem 3,

$$\begin{aligned} D^{\alpha_r+\pi} &> -\cos\alpha_r . A(z) - \sin\alpha_r . B(z) \\ &= \cos(\alpha_r+\pi)A(z) + \sin(\alpha_r+\pi)B(z) \end{aligned}$$

almost everywhere in $e_r$. Since (11.1) is true almost everywhere in $E$ it follows that $e_r$ is of measure zero, and so $\sum e_r$ is also of measure zero. That is, almost everywhere in $E$,

$$D^\alpha(z) = \cos\alpha . A(z) + \sin\alpha . B(z), \tag{11.2}$$

for $\alpha = \alpha_1, \alpha_2, \ldots$. We can now apply Lemma A of § 4 and deduce that (11.2) is true, for all values of $\alpha$, almost everywhere in $E$.

It follows at once that $A(z)$ and $B(z)$ are $D^0(z)$ and $D^{\frac{1}{2}\pi}(z)$ respectively. We obtain the final form of Theorem 4.

THEOREM 4B. *Under the conditions of Theorem 4A it is true that, almost everywhere in $E$, $D^0(z)$ and $D^{\frac{1}{2}\pi}(z)$ are finite and*

$$\lim_{\rho\to 0} \overline{\text{bound}}_{h\in S_\alpha(\rho,\eta)} \left\{\frac{f(z+h)-f(z)-kD^0(z)-lD^{\frac{1}{2}\pi}(z)}{|h|}\right\} = 0, \tag{11.3}$$

*for all values of $\alpha$ and $\eta$.*

At those points at which (11.3) is satisfied (and therefore (11.2) also) we shall say that there is an 'upper derivate plane'. We proceed to consider the values of $D^\alpha(z)$ at other points.

**12.** Theorem 4B implies that, at almost all points at which (11.3) is not satisfied, $D^\alpha(z) = +\infty$ for *some* value of $\alpha$. An argument similar to that of § 11 shows that at almost all these points $D^\alpha(z) = +\infty$ for *all* values of $\alpha$.

Suppose we take the set $\alpha_1, \alpha_2, \ldots$ considered above. We can prove as before that, for almost all values of $z$, *either* $D^{\alpha_r} = +\infty$ $(r = 1, 2, \ldots)$ *or* (11.3) is not satisfied. An application of Lemma A proves that, in the former case, $D^\alpha(z) = +\infty$ for all values of $\alpha$.

**13. Summary.** We have now proved that, given a measurable function $f(x, y)$, every point $(x, y)$, with the exception of a set of measure zero, belongs to one of two sets: (i) the set of points at which there is an upper derivate plane; (ii) the set of points at which $D^\alpha = +\infty$ for all values of $\alpha$.

Corresponding results can be obtained involving $D_\alpha$ and the 'lower derivate plane'—most simply by applying the arguments above to the function $\phi(x, y) = -f(x, y)$. We can summarize the combined results as follows:

Given a measurable function $f(x, y)$, at almost every point one of the following statements is true:

I. *There is an upper derivate plane and there is a lower derivate plane.* These are obviously identical: that is, *there is a unique tangent plane.* At these points $f(x, y)$ is totally differentiable in the sense of Stolz.

II A. *There is an upper derivate plane and* $D_\alpha = -\infty$ *for all values of* $\alpha$.

II B. *There is a lower derivate plane and* $D^\alpha = +\infty$ *for all values of* $\alpha$.

III. $D^\alpha = +\infty$ *and* $D_\alpha = -\infty$ *for all values of* $\alpha$.

The theorems of Rademacher and Stepanoff are corollaries of these results. If, in the notation quoted in § 1, $L_f(x, y)$ is finite, then only case I above is possible: therefore $f(x, y)$ is totally differentiable almost everywhere. We have, in fact, proved a deeper result; namely, that the necessary and sufficient condition that a measurable function be totally differentiable almost everywhere in a set $E$ is that, at almost all points of $E$, some $D^\alpha(z) < +\infty$ and some $D_\beta(z) > -\infty$ ($\alpha$ and $\beta$ need not be constant but may vary with $z$).

As a special case we deduce that all monotone functions of two variables are totally differentiable almost everywhere. Suppose, for

example, that $f(x, y)$ is an increasing function of $x$ and of $y$. Then $D^{\frac{1}{2}\pi} \leqslant 0$ and $D_{\frac{1}{2}\pi} \geqslant 0$: the result follows.

**14.** We conclude by considering certain easily-proved facts concerning the relation between the directed derivates defined in this paper and the ordinary partial derivates. We write, according to the usual definition,

$$D_x^+ f = \lim_{\rho \to 0} \overline{\text{bound}}_{0<k<\rho} \frac{f(x+k, y) - f(x, y)}{k},$$

with similar definitions for the seven other partial derivates at a point. It is immediately obvious that $D_x^+ \leqslant D^0$, and so on. At some points (for example, wherever there is a tangent plane) the two derivates are equal: we can show that equality occurs whenever the directed derivate is finite, except at a set of measure zero together with a set of points which is countable on every line $y =$ constant. If $D_x^+$ is measurable, this exceptional set is of plane measure zero: this is true if, for example, $f$ is a continuous function.

Suppose that $D_x^+ < D^0 < +\infty$ in a set $e$. By Theorem 3, $D^\pi = -D^0$ almost everywhere in $e$: and by a theorem due to Mrs. Young,* $D_x^- \leqslant D_x^+$ except at a set of points which is countable on every line $y =$ constant. Omitting these two exceptional sets, we have $D_x^- \leqslant -D^\pi$, which is false everywhere. Hence $e$ is entirely contained in the two exceptional sets. This is the result stated.

The present methods of investigation give no indication of the value of $D_x^+$ when $D^0 = +\infty$.

* G. C. Young, *Acta Math.* 37 (1914), 141–54 (144).

# ON VAN DER CORPUT'S METHOD AND THE ZETA-FUNCTION OF RIEMANN (III)

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**1.** It was proved by Hardy and Littlewood* that, as $T \to \infty$,

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\log T. \qquad (1)$$

Later, Ingham, using a method connected with the 'approximate functional equation' of Hardy and Littlewood, obtained the more precise result

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T-(1+\log 2\pi-2\gamma)T+O(T^{\frac{1}{2}}\log T), \quad (2)$$

where $\gamma$ is Euler's constant. As a further example of van der Corput's method, we shall now use this method to give an alternative proof of (2).

It is convenient to begin by proving two lemmas. We continue the system of numbering lemmas from the previous papers.† We shall not, however, have to refer to paper II, and we use the results of paper I as far as § 2.4 only.

**2.** Lemma 5. *If $f(x)$ satisfies the conditions of Lemma* 1, *and $\phi(x)$ is a real monotonic function, and $|\phi(x)| \leqslant M$, then*

$$\left|\sum_{a\leqslant n\leqslant b}\phi(n)\,e^{2\pi i f(n)}-\int_a^b \phi(x)\,e^{2\pi i f(x)}\,dx\right| < AM.$$

As in Lemma 1, we may take $a$ and $b$ to be integers; and it is sufficient to consider the case where $\phi(x)$ is steadily decreasing. Since by Lemma 1

$$\sum_{n=a}^{N}\left\{e^{2\pi i f(n)}-\int_n^{n+1} e^{2\pi i f(x)}\,dx\right\}$$

* For an account of known mean-value theorems see E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), Ch. II.

† *Quart. J. of Math.* (Oxford), 2 (1931), 161–73, and ibid., 313–20; the references to van der Corput's work are given at the end of I.

is bounded for $a \leqslant N < b$, it follows from Abel's partial summation formula that

$$\sum_{n=a}^{b-1} \phi(n)\left\{e^{2\pi i f(n)} - \int_n^{n+1} e^{2\pi i f(x)}\,dx\right\} = O(M). \tag{1}$$

Also

$$\left|\sum_{n=a}^{b-1} \int_n^{n+1} \{\phi(n)-\phi(x)\}e^{2\pi i f(x)}\,dx\right| \leqslant \sum_{n=a}^{b-1} \int_n^{n+1} \{\phi(n)-\phi(x)\}\,dx$$

$$\leqslant \sum_{n=a}^{b-1} \int_n^{n+1} \{\phi(n)-\phi(n+1)\}\,dx = \phi(a)-\phi(b) = O(M). \tag{2}$$

The result clearly follows from (1) and (2).

This lemma is closely connected with some much earlier work of Bromwich and Hardy,* but none of their results exactly covers it.

**3.** LEMMA 6. *If $f(x)$ and $\phi(x)$ are real functions, $f'(x)$ and $\phi(x)$ are monotonic, $f'(x) \geqslant m$ (or $\leqslant -m$), and $|\phi(x)| \leqslant M$, then*

$$\int_a^b \phi(x)\,e^{2\pi i f(x)}\,dx = O\left(\frac{M}{m}\right).$$

Particular cases of this lemma have already been used, for example in the proof of Lemma 2; it seems now worth while to state and prove the general case. By the second mean-value theorem,

$$\int_a^b \phi(x)\cos\{2\pi f(x)\}\,dx = \phi(a)\int_a^\xi \cos\{2\pi f(x)\}\,dx + \phi(b)\int_\xi^b \cos\{2\pi f(x)\}\,dx,$$

where $a < \xi < b$; and, if $f'(x) \geqslant m$,

$$\int_a^\xi \cos\{2\pi f(x)\}\,dx = \int_a^\xi \frac{2\pi f'(x)\cos\{2\pi f(x)\}}{2\pi f'(x)}\,dx$$

$$= \frac{\sin\{2\pi f(\eta)\}-\sin\{2\pi f(a)\}}{2\pi f'(a)} + \frac{\sin\{2\pi f(\xi)\}-\sin\{2\pi f(\eta)\}}{2\pi f'(\xi)},$$

where $a < \eta < \xi$; and a similar result holds for the integral over $(\xi, b)$. Hence

$$\int_a^b \phi(x)\cos\{2\pi f(x)\}\,dx = O\left(\frac{M}{m}\right),$$

and a similar result clearly holds for the sine integral.

* T. J. I'A. Bromwich, 'The relation between the convergence of series and integrals': *Proc. London Math. Soc.* (2) 6 (1908), 327–38; G. H. Hardy, 'Theorems connected with Maclaurin's test for the convergence of series': ibid. (2) 9 (1911), 126–44.

**4.** We have*

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<t} n^{-\frac{1}{2}-it} + O(t^{-\frac{1}{2}}) = Z(t) + O(t^{-\frac{1}{2}}),$$

say. It is sufficient to prove the required result with $Z(t)$ instead of $\zeta(\frac{1}{2}+it)$; for

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = \int_1^T |Z(t)|^2\,dt + O\Big\{\int_1^T |Z(t)|t^{-\frac{1}{2}}\,dt\Big\} + O\Big(\int_1^T t^{-1}\,dt\Big).$$

The last term is $O(\log T)$; and, if the required result holds for $Z(t)$, the middle term on the right is

$$O\Big[\Big\{\int_1^T |Z(t)|^2\,dt \int_1^T t^{-1}\,dt\Big\}^{\frac{1}{2}}\Big] = O\{(T\log T.\log T)^{\frac{1}{2}}\} = O(T^{\frac{1}{2}}\log T).$$

We therefore consider $Z(t)$. We have, writing $T_1 = \max(m,n)$,

$$\int_1^T |Z(t)|^2\,dt = \int_1^T \sum_{m<t} \frac{1}{m^{\frac{1}{2}+it}} \sum_{n<t} \frac{1}{n^{\frac{1}{2}-it}}\,dt = \sum_{m<T}\sum_{n<T} \frac{1}{(mn)^{\frac{1}{2}}} \int_{T_1}^T \Big(\frac{n}{m}\Big)^{it}\,dt$$

$$= \sum_{m<T} \frac{T-m}{m} + \frac{1}{i}\sum\sum_{m\neq n} \frac{(n/m)^{iT}}{(mn)^{\frac{1}{2}}\log(n/m)} - \frac{1}{i}\sum\sum_{m\neq n} \frac{(n/m)^{iT_1}}{(mn)^{\frac{1}{2}}\log(n/m)}$$

$$= \Sigma_1 - i\,\Sigma_2 + i\,\Sigma_3, \qquad (1)$$

say. A classical result gives

$$\Sigma_1 = T\log T + T(\gamma-1) + O(1). \qquad (2)$$

This gives the term $T\log T$, but not the required multiple of $T$; we shall see that multiples of $T$ arise from both $\Sigma_2$ and $\Sigma_3$.

**5.** Next consider $\Sigma_2$. We write

$$\Sigma_2 = \sum_{n\leqslant \frac{1}{2}m} + \sum_{\frac{1}{2}m<n<m} + \sum_{m<n<2m} + \sum_{2m\leqslant n}$$

$$= \Sigma_{21} + \Sigma_{22} + \Sigma_{23} + \Sigma_{24}. \qquad (1)$$

Then

$$\Sigma_{24} = \sum_{m<\frac{1}{2}T} \frac{m^{-iT}}{m^{\frac{1}{2}}} \sum_{2m\leqslant n<T} \frac{n^{iT}}{n^{\frac{1}{2}}\log(n/m)}.$$

By § 2.4 (2) of I

$$\sum_{n=2m}^{N} \frac{n^{iT}}{n^{\frac{1}{2}}} = O\Big(\frac{T^{\frac{1}{2}}}{m^{\frac{1}{2}}}\Big) \qquad (2m\leqslant N<T).$$

* See E. C. Titchmarsh, *The Zeta-function*, Th. 19.

Hence, by partial summation,

$$\sum_{2m\leqslant n<T} \frac{n^{iT}}{n^{\frac{1}{2}}\log(n/m)} = O\left(\frac{T^{\frac{1}{2}}}{m^{\frac{1}{2}}}\right).$$

Hence
$$\Sigma_{24} = \sum_{m<\frac{1}{4}T} O(T^{\frac{1}{2}}/m) = O(T^{\frac{1}{2}}\log T). \tag{2}$$

Similarly,
$$\Sigma_{21} = O(T^{\frac{1}{2}}\log T). \tag{3}$$

**6.** Putting $n = m+r$ in $\Sigma_{23}$, we obtain

$$\Sigma_{23} = \sum_{r<\frac{1}{2}T} \sum_{r<m<T-r} \frac{\{(m+r)/m\}^{iT}}{\{m(m+r)\}^{\frac{1}{2}}\log\{(m+r)/m\}}. \tag{1}$$

Let

$$f(x) = \frac{T}{2\pi}\log\frac{x+r}{x}, \qquad f'(x) = -\frac{Tr}{2\pi x(x+r)}, \qquad f''(x) = \frac{T}{2\pi}\frac{2xr+r^2}{x^2(x+r)^2}.$$

Then, for $r < x$,
$$ATrx^{-3} < f''(x) < A'Trx^{-3}.$$

Hence by Theorem 1, if $M' \leqslant 2M$,

$$\begin{aligned}\sum_{m=M}^{M'} e^{2\pi i f(m)} &= O\{M(TrM^{-3})^{\frac{1}{2}}\}+O\{(TrM^{-3})^{-\frac{1}{2}}\}\\ &= O(M^{-\frac{1}{2}}T^{\frac{1}{2}}r^{\frac{1}{2}})+O(T^{-\frac{1}{2}}r^{-\frac{1}{2}}M^{\frac{3}{2}}).\end{aligned}$$

Denoting the general term in (1) by $\psi(m,r)$, it follows by partial summation that

$$\sum_{m=M}^{M'} \psi(m,r) = O(M^{-\frac{1}{2}}T^{\frac{1}{2}}r^{-\frac{1}{2}})+O(M^{\frac{3}{2}}T^{-\frac{1}{2}}r^{-\frac{3}{2}}).$$

Adding sums of this type in the usual way, we obtain

$$\begin{aligned}\sum_{r<m\leqslant\sqrt{(Tr)}} \psi(m,r) &= O(r^{-\frac{1}{2}}T^{\frac{1}{2}}r^{-\frac{1}{2}})+O[\{\sqrt{(Tr)}\}^{\frac{3}{2}}T^{-\frac{1}{2}}r^{-\frac{3}{2}}]\\ &= O(T^{\frac{1}{2}}r^{-1})+O(T^{\frac{1}{2}}r^{-\frac{1}{2}}),\end{aligned}$$

and the sum of these terms over $r < \frac{1}{2}T$ is $O(T^{\frac{1}{2}}\log T)$.

For $m \geqslant \sqrt{(Tr)}$, we have

$$|f'(m)| < \tfrac{1}{2}Tr/(\pi m^2) < \tfrac{1}{2},$$

and we can use Lemmas 5 and 6. Lemma 5 gives

$$\sum_{\sqrt{(Tr)}\leqslant m<T-r} \psi(m,r) = \int_{\sqrt{(Tr)}}^{T-r} \psi(x,r)\,dx + O\left(\frac{1}{r}\right).$$

The $O$-term contributes $O(\log T)$ on summing. The integral is $O(Tr^{-2})$, by Lemma 6, since $|f'(x)| > Ar/T$; and

$$\sum_{\sqrt{T}\leqslant r<\frac{1}{2}T} O(Tr^{-2}) = O(T^{\frac{1}{2}}).$$

We have still to consider the sum over $r < \sqrt{T}$. Putting

$$T \log \frac{x+r}{x} = u, \qquad x = \frac{r}{e^{u/T}-1},$$

we obtain

$$\int_{\sqrt{(Tr)}}^{T-r} \psi(x,r)\,dx = \int_{u_1}^{u_2} \frac{e^{\frac{1}{2}u/T}}{e^{u/T}-1} \frac{e^{iu}}{u}\,du,$$

where $u_1 = T\log\{T/(T-r)\}$, $u_2 = T\log(1+r^{\frac{1}{2}}T^{-\frac{1}{2}})$.

We can replace this by

$$T\int_{u_1}^{u_2} \frac{e^{iu}}{u^2}\,du,$$

with error

$$\int_{u_1}^{u_2} \left\{\frac{e^{\frac{1}{2}u/T}}{e^{u/T}-1} - \frac{T}{u}\right\}\frac{e^{iu}}{u}\,du = \int_{u_1}^{u_2} O\left(\frac{1}{u}\right)du = O(\log T),$$

which gives $O(T^{\frac{1}{2}}\log T)$ on summing. Next, we can replace $u_2$ by $\infty$, with error

$$T\int_{u_2}^{\infty} \frac{e^{iu}}{u^2}\,du = O\left(\frac{T}{u_2^2}\right) = O\left(\frac{1}{r}\right),$$

which gives $O(\log T)$ on summing; and we can replace $u_1$ by $r$, with error

$$T\int_{r}^{u_1} \frac{e^{iu}}{u^2}\,du = O\left\{\frac{T(u_1-r)}{r^2}\right\} = O(1),$$

which gives $O(T^{\frac{1}{2}})$ on summing. The sum thus obtained is

$$T\sum_{r<\sqrt{T}}\int_{r}^{\infty} \frac{e^{iu}}{u^2}\,du = cT + T\sum_{r\geqslant\sqrt{T}} O\left(\frac{1}{r^2}\right) = cT + O(T^{\frac{1}{2}}),$$

where

$$c = \sum_{r=1}^{\infty}\int_{r}^{\infty} \frac{e^{iu}}{u^2}\,du.$$

Since $\Sigma_{22}$ is minus the conjugate of $\Sigma_{23}$, the two together give

$$\Sigma_{22} + \Sigma_{23} = (c-\bar{c})T + O(T^{\frac{1}{2}}\log T), \tag{1}$$

where

$$c-\bar{c} = 2i\sum_{r=1}^{\infty}\int_{r}^{\infty} \frac{\sin u}{u^2}\,du.$$

The sum of this series may be derived from the formula*

$$\zeta(s)+\frac{x^{1-s}}{1-s}+\left(\tfrac{1}{2}-x+[x]\right)x^{-s}-\sum_{n\leqslant x}n^{-s}=2s(2\pi)^{s-1}\sum_{n=1}^{\infty}n^{s-1}\int_{2n\pi x}^{\infty}\frac{\sin u}{u^{s+1}}\,du.$$

Putting $x=1/2\pi$, making $s\to 1$, and observing that

$$\lim_{s\to 1}\left\{\zeta(s)-\frac{1}{s-1}\right\}=\gamma,$$

we obtain

$$c-\bar{c}=i(\gamma-\log 2\pi+\pi-1). \tag{2}$$

**7.** Now consider $\sum_3$. Dividing it up in the same way as $\sum_2$, and using a corresponding notation, we have

$$\Sigma_{34}=\sum_{m<\frac{1}{4}T}\frac{1}{m^{\frac{1}{2}}}\sum_{2m\leqslant n<T}\frac{(n/m)^{in}}{n^{\frac{1}{2}}\log(n/m)}.$$

Let

$$f(x)=\frac{x}{2\pi}\log\frac{x}{m},\qquad f'(x)=\frac{1}{2\pi}\left(\log\frac{x}{m}+1\right),\qquad f''(x)=\frac{1}{2\pi x}.$$

Then Theorem 1 gives

$$\sum_{n=N}^{N'}e^{2\pi i f(n)}=O(N^{\frac{1}{2}})\qquad(N'\leqslant 2N).$$

Hence, by partial summation,

$$\sum_{n=N}^{N'}\frac{(n/m)^{in}}{n^{\frac{1}{2}}\log(n/m)}=O(1),$$

and, adding sums of this type in the usual way,

$$\sum_{2m\leqslant n<T}\frac{(n/m)^{in}}{n^{\frac{1}{2}}\log(n/m)}=O(\log T).$$

Hence

$$\Sigma_{34}=\sum_{m<\frac{1}{4}T}O(m^{-\frac{1}{2}}\log T)=O(T^{\frac{1}{2}}\log T). \tag{1}$$

Similarly,

$$\Sigma_{31}=O(T^{\frac{1}{2}}\log T). \tag{2}$$

**8.** Putting $n=m+r$ in $\sum_{33}$, we obtain

$$\Sigma_{33}=\sum_{r<\frac{1}{4}T}\sum_{r<m<T-r}\frac{\{(m+r)/m\}^{i(m+r)}}{\{m(m+r)\}^{\frac{1}{2}}\log\{(m+r)/m\}}. \tag{1}$$

Let

$$f(x)=\frac{x+r}{2\pi}\log\frac{x+r}{x},\qquad f'(x)=\frac{1}{2\pi}\left(\log\frac{x+r}{x}-\frac{r}{x}\right).$$

* G. H. Hardy and J. E. Littlewood, 'The zeros of Riemann's zeta-function on the critical line': *Math. Zeitschrift*, 10 (1921), 283–317, Lemma 1.

Then $$|f'(x)| < \tfrac{1}{4}r^2/(\pi x^2) < \tfrac{1}{2} \qquad (x > r).$$

Hence, by Lemma 5,

$$\sum_{r<m<T-r} \frac{\{(m+r)/m\}^{i(m+r)}}{\{m(m+r)\}^{\frac{1}{4}}\log\{(m+r)/m\}} = \int_{r+1}^{T-r} \frac{\{(x+r)/x\}^{i(x+r)}}{\{x(x+r)\}^{\frac{1}{4}}\log\{(x+r)/x\}}\,dx + O\left(\frac{1}{r}\right), \quad (2)$$

and the $O$-term gives $O(\log T)$ on summing. In the integral we put $(x+r)\log\{(x+r)/x\} = u$, and obtain

$$\int \frac{e^{iu}\,du}{\{x(x+r)\}^{\frac{1}{4}}\log\{(x+r)/x\}[r/x - \log\{(x+r)/x\}]}.$$

We replace this integral by

$$\int \frac{re^{iu}\,du}{2[(x+r)\log\{(x+r)/x\} - r]^2} = \int \frac{re^{iu}}{2(u-r)^2}\,du.$$

The difference between the two integrals is of the form

$$\frac{1}{r}\int \phi\left(\frac{r}{x}\right)e^{iu}\,du,$$

where $\phi(z)$, having a simple pole at $z = 0$, is steadily decreasing and of the form $O(1/z)$ in some interval $0 < z \leqslant a$. The corresponding integral is

$$\frac{1}{r}\int_{T\log\{T/(T-r)\}}^{r(1+1/a)\log(1+a)} \phi\left(\frac{r}{x}\right)e^{iu}\,du = O\left\{\frac{1}{r}\phi\left(\frac{r}{T-r}\right)\right\} = O\left(\frac{T}{r^2}\right),$$

and it is also

$$O\left\{\frac{1}{r}\int_{T\log\{T/(T-r)\}}^{r(1+1/a)\log(1+a)} \frac{x}{r}\,du\right\} = O\left\{\frac{1}{r^2}\int_{r/a}^{T-r} x\left|\frac{du}{dx}\right|dx\right\} = O\left\{\int_{r/a}^{T-r}\frac{dx}{x}\right\} = O(\log T).$$

Summing with respect to $r$, we therefore obtain

$$\sum_{r<\sqrt{T}} O(\log T) + \sum_{\sqrt{T}\leqslant r<\frac{1}{4}T} O(Tr^{-2}) = O(T^{\frac{1}{2}}\log T) + O(T^{\frac{1}{2}}) = O(T^{\frac{1}{2}}\log T).$$

If $a < 1$, the part of the integral in (2) with $x < r/a$ is $O(1/r)$, by Lemma 6, so that this part contributes $O(\log T)$ on summing. Hence

$$\Sigma_{33} = \tfrac{1}{2}\sum_{r<\frac{1}{4}T} r \int_{T\log\{T/(T-r)\}}^{Ar} \frac{e^{iu}}{(u-r)^2}\,du + O(T^{\frac{1}{2}}\log T), \quad (3)$$

where $A > 1$. We can replace the upper limit by $\infty$, with error

$$\sum_{r<\frac{1}{2}T} r \int_{Ar}^{\infty} \frac{e^{iu}\,du}{(u-r)^2} = \sum_{r<\frac{1}{2}T} O\left(\frac{1}{r}\right) = O(\log T).$$

We have then to consider

$$\sum_{r<\frac{1}{2}T} r \int_{T\log\{T/(T-r)\}}^{\infty} \frac{e^{iu}\,du}{(u-r)^2} = \sum_{r<\frac{1}{2}T} re^{ir} \int_{v_r}^{\infty} \frac{e^{iv}}{v^2}\,dv, \tag{4}$$

where

$$v_r = T\log\{T/(T-r)\} - r.$$

Let

$$s_n = \sum_{r=1}^{n} re^{ir},$$

so that $s_n = O(n)$. If $R$ is the greatest value of $r$ in the sum, we obtain

$$\sum_{r=1}^{R} (s_r - s_{r-1}) \int_{v_r}^{\infty} \frac{e^{iv}}{v^2}\,dv = \sum_{r=1}^{R-1} s_r \int_{v_r}^{v_{r+1}} \frac{e^{iv}}{v^2}\,dv + s_R \int_{v_R}^{\infty} \frac{e^{iv}}{v^2}\,dv.$$

The last term is $O(Rv_R^{-2}) = O(1)$. In the sum we put $e^{iv} = 1 + O(v)$ for $r < \sqrt{T}$, and $e^{iv} = O(1)$ for $r \geqslant \sqrt{T}$, and obtain

$$\sum_{r<\sqrt{T}} s_r \int_{v_r}^{v_{r+1}} \left\{\frac{1}{v^2} + O\left(\frac{1}{v}\right)\right\} dv + \sum_{\sqrt{T}\leqslant r<\frac{1}{2}T} s_r \int_{v_r}^{v_{r+1}} O\left(\frac{1}{v^2}\right) dv$$

$$= \sum_{r<\sqrt{T}} s_r \left\{\left(\frac{1}{v_r} - \frac{1}{v_{r+1}}\right) + O\left(\frac{v_{r+1}-v_r}{v_r}\right)\right\} + \sum_{\sqrt{T}\leqslant r<\frac{1}{2}T} s_r\, O\left(\frac{v_{r+1}-v_r}{v_r^2}\right).$$

Since $v_r > Ar^2/T$, and $v_{r+1} - v_r = O(r/T)$, the $O$-terms are

$$\sum_{r<\sqrt{T}} O(1) + \sum_{\sqrt{T}\leqslant r<\frac{1}{2}T} O(T/r^2) = O(T^{\frac{1}{2}}).$$

Also

$$\sum_{r<\sqrt{T}} s_r \left(\frac{1}{v_r} - \frac{1}{v_{r+1}}\right) = \sum_{r<\sqrt{T}} \frac{s_r - s_{r-1}}{v_r} + O(T^{\frac{1}{2}}) = \sum_{r<\sqrt{T}} \frac{re^{ir}}{v_r} + O(T^{\frac{1}{2}}).$$

Now

$$\frac{r}{v_r} = \frac{2T}{r} + w_r,$$

where $w_r$ is monotonic and bounded. Hence

$$\sum_{r<\sqrt{T}} \frac{re^{ir}}{v_r} = 2T \sum_{r<\sqrt{T}} \frac{e^{ir}}{r} + O(1) = 2T \sum_{r=1}^{\infty} \frac{e^{ir}}{r} + O(T^{\frac{1}{2}}).$$

Hence by (3), and the corresponding formula for $\sum_{32}$,

$$\sum_{32}+\sum_{33}=2iT\sum_{r=1}^{\infty}\frac{\sin r}{r}+O(T^{\frac{1}{2}}\log T)=iT(\pi-1)+O(T^{\frac{1}{2}}\log T). \quad (5)$$

The result now follows from (1) and (2) of § 4, (2) and (3) of § 5, (1) and (2) of § 6, (1) and (2) of § 7, and (5) above.

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*Correction.* In note I of this series, p. 171, lines 7 and 6 from the bottom, for $f^{(4)}(\nu)$ read $f^{(5)}(\nu)$, and for 'generally' read 'for $k \leqslant 4$'. I find that the statement is false for $k = 5$ and a certain value of $\mu/\nu$. Since we actually take $k = 4$, the proof remains valid; but it would be more illuminating to follow the method of Landau's *Vorlesungen*, ii. 270–1, where this difficulty is avoided.

# A CLASS OF TRIANGLE-FUNCTIONS (I).

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SUPPOSE a variable $u$ is subjected to the group of linear transformations associated, in the Schwarzian theory of conformal representation, with a triangle whose contour consists of straight lines or circular arcs and whose angles are $\pi/m$, $\pi/n$, $\pi/p$, where $m$, $n$, $p$ are positive integers greater than unity, zero angles being included. I propose to determine the most general one-valued function $w(u)$ which has the property

$$w(u') = \frac{Aw(u)+B}{Cw(u)+D},$$

where $A$, $B$, $C$, $D$ are constants, and $u'$ is any linear transformation of $u$ from its group. It is necessary to discuss the earlier question what limitations must be placed on the group of $w$; this proves indeed to be the whole question.

The classical instances which have led me to my present inquiry are found in Klein's expressions of the polyhedral irrationalities as modular functions. Other examples are the functions given by Kluyver to effect the conformal representations of the areas exterior to certain triangles.*

I am indebted to Professor A. C. Dixon for pointing out that my problem, slightly restated, is included, as a very special case, in Poincaré's theory of Zeta-Fuchsian functions.† Suppose the $u$-triangle is mapped upon the half-plane of a variable $z$,‡ the $z$-points 0, 1, $\infty$ corresponding to the vertices of the triangle. We know that $z$ is a one-valued function of $u$, and that descriptions of circuits encircling the $z$-points 0, 1 are equivalent to linear transformations of $u$ which generate the whole group of $u$. Suppose further that $y_1$, $y_2$ are functions of $z$ with branch-points at 0, 1, $\infty$, and that their final values $y_1$, $y_2$ after the description of such a circuit are given by

$$y_1' = Ay_1+By_2$$
$$y_2' = Cy_1+Dy_2.$$

* F. Klein, *Vorlesungen über das Ikosaeder* (1884), 131–3. J. C. Kluyver, *Zeitschrift der Math. u. Phys.* 40 (1895), 129–50.

† H. Poincaré, *Acta Mathematica*, 5 (1884), 209–78.

‡ The symbol $z$ will bear this meaning throughout this paper.

It is clear that we obtain a function of the type sought if we write $w(u) = y_1/y_2$, provided that this function is one-valued (this is, of course, adequately discussed in Poincaré's memoir). But we should equally satisfy the conditions of the problem, as I have stated it, if the corresponding laws for $y_1'$, $y_2'$ were of the form

$$y_1' = (Ay_1 + By_2)P(z)$$
$$y_2' = (Cy_1 + Dy_2)P(z),$$

where $P(z)$ is any function of $z$. I shall, in fact, find that every solution of the problem is included in those which are readily constructed after reference to the writings of Klein and Poincaré.

The problem is immediately reduced, when a possible group of $w$ has been assigned, to that of finding three particular functions $w_1$, $w_2$, $w_3$ which are subject to the same group. Suppose that, as a result of applying a linear transformation to $u$, $w$ is transformed into $w'$, where $w' = (Aw+B)/(Cw+D)$, and that $w_1$, $w_2$, $w_3$ are similarly transformed into $w_1'$, $w_2'$, $w_3'$. Then

$$\frac{(w'-w_1')(w_2'-w_3')}{(w'-w_3')(w_2'-w_1')} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)},$$

i.e. the cross-ratio of the functions $w$, $w_1$, $w_2$, $w_3$ is unaltered by the operation of any transformation from the group of $u$. But, according to the general theory, any one-valued function automorphic under the group of transformations of $u$ is a one-valued function of $z$, so that we may write

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = f(z).$$

*Restrictions to be placed on the group of $w$.* In the first instance I suppose that none of the angles $\pi/m$, $\pi/n$, $\pi/p$ of the $u$-triangle $ABC$ is zero (i.e. infinite values of $m$, $n$, $p$ are excluded).

The group of $u$ is generated by any pair of three linear transformations $A(u)$, $B(u)$, $C(u)$. $A(u)$ is an elliptic substitution with multiplier $\exp(2i\pi/m)$ and fixed points $A$, $A'$, where $A'$ is the second intersection of the circles $BA$, $AC$. It is the analytic equivalent of successive inversions in these circles taken in this order. $B(u)$, $C(u)$ have, of course, similar meanings and it is evident that $ABC(u) \equiv u$. [The convention as to the meaning of $ST$ varies; in this paper $ST(u) \equiv S\{T(u)\}$.] Suppose the corresponding transformations of $w$ are $A'(w)$, $B'(w)$, $C'(w)$. Since $w$ is a one-valued function of $u$, and

$ABC(u) \equiv u$, it follows that $A'B'C'(w) \equiv w$. Since also $A^m(u) \equiv u$, $A'^m(w) \equiv w$, for the same reason, i.e. the multiplier of $A'(w)$ is $\exp(2im'\pi/m)$, where $m'$ is an integer and $0 \leqslant m' < m$. This requires that $A'(w)$ have two fixed points.

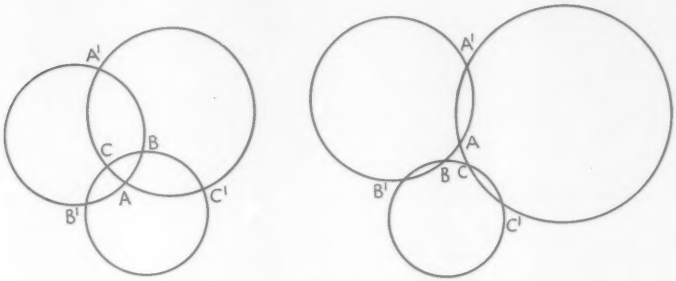


FIG. 1

Now consider the fixed points of the $w$-transformations. There are three possibilities:

(i) the fixed points of $B'(w)$, and therefore of $C'(w)$ by reason of the identity $A'B'C'(w) \equiv w$, coincide with those of $A'(w)$;

(ii) one fixed point of $B'(w)$, and therefore one fixed point of $C'(w)$, coincides with one fixed point of $A'(w)$, and the remaining fixed points are distinct from these and from each other;

(iii) all the fixed points are distinct.

(i) If the fixed points of $A'(w)$, $B'(w)$, $C'(w)$ coincide, we may suppose that a suitable linear transformation of the variable $w$ has been made, so that one fixed point is zero and one is infinity. Then

$$A'(w) \equiv w\exp(2im'\pi/m),\ B'(w) \equiv w\exp(2in'\pi/n),$$

$$C'(w) \equiv w\exp(2ip'\pi/p),$$

and, since $A'B'C'(w) \equiv w$,

$$\frac{m'}{m}+\frac{n'}{n}+\frac{p'}{p} = 0, 1, 2.$$

The only way of obtaining zero for the right-hand member of this equation is by taking $m'$, $n'$, $p'$ all zero, and $w$ is then automorphic for all transformations of the group of $u$, i.e. $w$ is a one-valued function of $z$.

It is clear that no solutions of the equations

$$\frac{m'}{m}+\frac{n'}{n}+\frac{p'}{p}=1,2$$

can be found if $m$, $n$, $p$ are relatively prime. When solutions do exist, the construction of the functions we seek is trivial. If $w$, $w_1$ are two such functions with the same multipliers, then clearly $w=w_1 f(z)$. Now if, as is conventionally arranged, the $z$-points $0$, $1$, $\infty$ correspond to the $u$-points $A$, $B$, $C$, we may take $w_1$ to be $z^{m'/m}(z-1)^{n'/n}$, for we know that, near $z=0,1$,

$$u-a \sim z^{1/m}; \qquad u-b \sim (z-1)^{1/n}.$$

These trivial functions (with the slight modifications possible when some of the angles $A$, $B$, $C$ are zero) are the only analogues in this theory of Appell's functions with constant multipliers. We shall consider groups of this type no further.

(ii) When $A'(w)$, $B'(w)$, $C'(w)$ have one fixed point in common, we suppose that this fixed point has been removed to infinity by a previous linear transformation of $w$. The transformations $A'(w)$, $B'(w)$, $C'(w)$ are now equivalent to rotations in the plane of $w$ about the finite fixed points $A$, $B$, $C$ of amounts $2m'\pi/m$, $2n'\pi/n$, $2p'\pi/p$, and it is again necessary that

$$\frac{m'}{m}+\frac{n'}{n}+\frac{p'}{p}=1,2.$$

Suppose now that the fixed points $B$, $C$ are given in position in the $w$-plane. We join $BC$, and make the angles $CBA_1$, $BCA_1$ equal to $m'\pi/m$, $n'\pi/n$. Then $B'(w)$ is equivalent to successive reflections in $CB$, $BA_1$, and $C'(w)$ to successive reflections in $A_1C$, $CB$, i.e. $B'C'(w)$ to successive reflections in $A_1C$, $BA_1$. But $A'B'C'(w)\equiv w$, so that $A'(w)$ is equivalent to successive reflections in $BA_1$, $A_1C$, i.e. $A_1$ must be the fixed point $A$, and the angle $CAB$ may be $m'\pi/m$ or $(m+m')\pi/m$, either of which leads to the multiplier $\exp(2im'\pi/m)$.

If the right-hand member of the equation above is 1, the angle $BAC$ is $m'\pi/m$, and the triangle $ABC$ is a rectilinear triangle in the ordinary sense, described in the counter-clockwise direction.

If, however, the right-hand member is 2, the figure we obtain is such as is illustrated in fig. 2. $BC$ is joined in the sense that the side goes to infinity and returns from the opposite direction, and the triangle is the rectilinear figure of infinite area which lies to the left

as the sides are described in the sense $CAB\Omega C$. We observe that the angle $BAC$ in this figure is not $m'\pi/m$ but $(m+m')\pi/m$.

FIG. 2

In either case the group of $w$ is that associated with the conformal representation of the triangle upon the $z$ half-plane. It is to be noted that, as we do not require $z$ to be a one-valued function of $w$, the limitation applied to the triangle in the $u$-plane, that its angles are sub-multiples of $\pi$, does not apply to the $w$-triangle.

[There are, of course, generalized triangles of other types than the one I have described. It is not necessary to consider them here. The group of $w$ must be one of the two types I have enumerated, and it will be shown subsequently that it is only necessary to find one triangle with which the group may be associated.]

We note further that, if $p'$ were zero, $C'(w)$ would be the identical transformation. But $C'(w)$ is equivalent to successive reflections in $AC$, $CB$ so that these lines would coincide, $A$, $B$, $C$ would be collinear, and $m'$, $n'$ would also be zero, and we should return again to the class of functions automorphic under the group of $u$.

(iii) When the fixed points of $A'(w)$, $B'(w)$, $C'(w)$ are distinct, we suppose a linear transformation applied to the plane so that one fixed point of $B'(w)$ is zero and the other infinity, so that

$$B'(w) \equiv w\exp(2in'\pi/n).$$

Let $A'(w)$ be given in the form

$$\frac{w'-\alpha}{w'-\beta} = \exp(2im'\pi/m)\,\frac{w-\alpha}{w-\beta}.$$

Since $C'(w)$ is the inverse of $A'B'(w)$, $C'(w)$ has the form

$$\frac{w-\alpha}{w-\beta} = \exp(2im'\pi/m)\,\frac{w'\exp(2in'\pi/n)-\alpha}{w'\exp(2in'\pi/n)-\beta}.$$

Now the multiplier of the transformation

$$w' = \frac{aw+b}{cw+d}$$

is given by the formula

$$\frac{(K+1)^2}{K} = \frac{(a+d)^2}{ad-bc}.$$

Hence the multiplier of $C'(w)$ is given by

$$(\alpha-\beta)(K^{\frac{1}{2}}+K^{-\frac{1}{2}}) = 2\alpha\cos\left(\frac{m'}{m}-\frac{n'}{n}\right)\pi - 2\beta\cos\left(\frac{m'}{m}+\frac{n'}{n}\right)\pi.$$

Hence the ratio $\alpha:\beta$ is real, i.e. $\alpha$, $\beta$ are collinear with zero, so that before the preliminary transformation was made, the fixed points $A, A', B, B'$ were concyclic. Thus $A, A'$; $B, B'$; $C, C'$ are the three pairs of intersections of three circles, and it follows, by a repetition of the argument of (ii) above, that the angles of the triangle $ABC$ are $m'\pi/m$, $n'\pi/n$, $p'\pi/p$, and that the group of $w$ is that associated with the conformal representation of this triangle upon the $z$ half-plane.

For completeness it must be shown that such a triangle exists for any values $m'$, $n'$, $p'$ subject to the limitations $0 < m' < m$; $0 < n' < n$; $0 < p' < p$. [The impossibility of zero values of $m'$, $n'$, $p'$ can be shown as before.]

The demonstration is shortened by recalling a simplification used by Schwarz in his fundamental memoir on the representation of a curvilinear triangle. Amongst the triangles determined by three circles intersecting in three pairs of points $A, A'$; $B, B'$; $C, C'$ there are eight which may be associated in four pairs, $ABC, A'B'C'$; $A'BC, AB'C'$; $AB'C, A'BC'$; $ABC', A'B'C$ (see fig. 1). The triangles of each pair have the same angles, the angles of the four pairs being $A, B, C$; $A, \pi-B, \pi-C$; $\pi-A, B, \pi-C$; $\pi-A, \pi-B, C$. It is evident from the figure that, if the contours of the triangles are described in the senses indicated by the orders of the letters, in one of each pair the area bounded by the contour is to the left, in the other to the right. We consider the four in which the area is to the left. Since the circles occur in the same order as the contours are described, the groups of these four triangles, considered as generated by successive inversions in the sides, are the same. Hence, by selecting the right triangle of the four, we may restrict ourselves to showing that the $w$-triangle exists when at least two of the inequalities, $m' \leqslant \frac{1}{2}m$, $n' \leqslant \frac{1}{2}n$, $p' \leqslant \frac{1}{2}p$, hold.

We require, then, to show the existence of a curvilinear triangle of given angles $A$, $B$, $C$ where $B$, $C$ are not obtuse, but $A$ may be.

We take the general case, $B$ less than $C$, and both acute, the remaining cases being readily filled in.

We may suppose the arms of the angle $B$ to be straight. On one of them take a point $C$, and draw a straight line, making an angle equal to the required angle $C$, and meeting the other arm of $B$ at $P$. Then, since $PB$ is greater than $PC$, a circle, with centre $P$ and radius $PC$, cuts $PB$ twice on the same side of $B$ as $P$, say at $Q$, $R$. Circles touching $PC$ at $C$ and passing through $Q$, $R$ touch $PB$, i.e. the curvilinear angle $BQC$ is zero and the angle $BRC$ is $\pi$. If $A$ is allowed

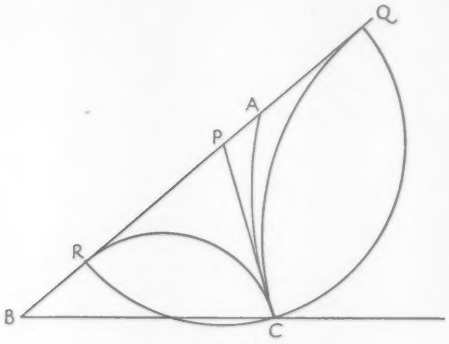


FIG. 3

to move continuously from $Q$ to $R$, and a circle is drawn touching $PC$ at $C$ and passing through $A$, the curvilinear angle $BAC$ will vary continuously between zero and $\pi$. Hence a triangle of the required angles exists.

In all that precedes the possibility of zero angles in the $u$-triangle $ABC$ has been excluded. If, however, the angle $C$ is zero, the transformation $C(u)$ is parabolic, and it is impossible to restore the initial value of $u$ by any number of repetitions of $C(u)$. There is no compulsion on $C'(w)$ to be elliptic. The modifications of our results required are not serious. They may be summarized briefly.

In any case, at least as many of the three transformations $A'(w)$, $B'(w)$, $C'(w)$ must be elliptic as there are non-zero angles in the $u$-triangle. To this we have only to add to the discussion of (i) above that the product of the multipliers $K_1$, $K_2$, $K_3$ of $A'(w)$, $B'(w)$, $C'(w)$ must be unity. The amendment of (ii) is almost equally simple. Either $K_1K_2K_3 = 1$ or $K_1K_2 = 1$ and the third transformation is

parabolic, or all three transformations are parabolic with the same fixed point.

The amendment of (iii) is a little longer. The earlier argument breaks down at the point where it was shown that $A$, $A'$, $B$, $B'$ were concyclic. The argument, however, can be modified to show that, the multipliers $K_1$, $K_2$, $K_3$, the two fixed points of $A'(w)$, and one fixed point of $B'(w)$ being given, the other fixed point of $B'(w)$ is determined. But three assigned points can be transformed into any other set of three by a linear transformation so that the group is essentially determined by the three multipliers.

If, however, two of the transformations have multipliers $K_1$, $K_2$, and the third is parabolic, the group is determined by the numbers $K_1$, $K_2$.

If one of the transformations has a multiplier $K_1$, and the other two are parabolic, the group is determined by the number $K_1$.

If all the transformations are parabolic, the group is determined.

Of these statements only the last will be proved explicitly. By a preliminary transformation we may suppose that the fixed points of $A'(w)$, $B'(w)$ have been transferred to zero and infinity. We may thus write $A'(w)$ in the form

$$\frac{1}{w'} = \frac{1}{w} + K_1',$$

and $B'(w)$ in the form $\qquad w' = w + K_2'.$

This gives $C'(w)$ in the form

$$\frac{1}{w} = \frac{1}{w' + K_2'} + K_1',$$

and, in order that $C'(w)$ may be parabolic, it is necessary that $K_1' K_2' + 4 = 0$. [This appears to show that two, $K_1'$, $K_2'$, of the additive constants of the transformations must be equal, which is absurd, for, of course, $A'(w)$, $B'(w)$, $C'(w)$ could be permuted in the preceding discussion in any manner. It must be remembered that the additive constant of a parabolic transformation, unlike the multipliers of the other types, is not invariant under linear transformation; e.g. if the transformation $w' = w + 2$ is transformed by the substitution $w' = 2W'$, $w = 2W$, it becomes $W' = W + 1$.]

It will be shown in the next section that functions of all the types indicated are constructed by means of hypergeometric functions of $z$. What the preceding argument is designed to show is that, although

*a priori* it was conceivable that other functions existed, they do not exist in fact.

*Construction of the functions.* We shall deal first with the general case of (iii) above when the $u$-triangle has non-zero angles. We know, from the Schwarzian theory, that we may take $w_1$ as the ratio of two solutions $y_1$, $y_2$ of the hypergeometric equation

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma'-(\alpha'+\beta'+1)z\}\frac{dy}{dz}-\alpha'\beta'y=0.$$

where

$$\left(\frac{m'}{m}\right)^2=(1-\gamma')^2, \qquad \left(\frac{n'}{n}\right)^2=(\gamma'-\alpha'-\beta')^2; \qquad \left(\frac{p'}{p}\right)^2=(\alpha'-\beta')^2.$$

Quite plainly we may take $w_2$ to be the ratio of their second and $w_3$ to be the ratio of their first derivatives. The second derivatives $y_1''$, $y_2''$ may now be eliminated from the fraction

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

and we arrive at the result

$$w\left\{\frac{dy_2}{dz}-y_2f(z)\right\}=\left\{\frac{dy_1}{dz}-y_1f(z)\right\}.$$

This is strictly in accordance with the result of Poincaré* as was to be expected.

The modifications for the other cases are obvious. When the group of $w$ is that associated with a rectilinear figure, a non-essential change may be introduced by writing $w_1$ in the form of an Abelian integral as given by the Schwarz-Christoffel formula.

When the $u$-triangle has zero angles, we have merely to replace $m'/m$, $n'/n$, $p'/p$ in the formulae determining $\alpha'$, $\beta'$, $\gamma'$ by $(\log K_1)/2i\pi$, $(\log K_2)/2i\pi$, $(\log K_3)/2i\pi$, if the function required has transformations with multipliers; for parabolic transformations we replace the corresponding numbers by zero. In particular, if the $u$-triangle has three zero angles, we may have three parabolic transformations of $w$. In this case the groups of $u$ and $w$ are identical, and $w_1$ may be $u$ itself. But we know that, if the $u$-triangle is the triangle of three zero angles in the elliptic modular configuration as usually drawn,

$$u=\frac{iK'}{K}, \qquad z=k^2=c,$$

* p. 230.

in the Jacobian notations, and we find the most general function which undergoes the same transformations as $u$, when this variable is subjected to those elliptic modular transformations which leave $c$ unaltered, to be, in its most compact form,

$$i\{K'-E'f(c)\}/\{K-(K-E)f(c)\}.$$

The simple evaluation of $w$ suffers from a serious defect, common to the whole general theory of triangle-functions; it is expressed explicitly in terms of $z$. But $z$ itself is only expressed implicitly as a function of $u$; in fact, $u$ is the ratio of two solutions of the equation

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{dy}{dz}-\alpha\beta y=0,$$

where $\quad \frac{1}{m^2}=(1-\gamma)^2, \qquad \frac{1}{n^2}=(\gamma-\alpha-\beta)^2, \qquad \frac{1}{p^2}(\alpha-\beta)^2.$

In a certain number of special cases, however, the reversal of the relation between $u$, $z$ has been carried out, and these offer scope for explicit evaluations of $w$. I propose to return to such evaluations in a subsequent paper.

# ON A PROBLEM IN THE ADDITIVE THEORY OF NUMBERS (V)

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## 1. Introductory

In previous papers* we have discussed the representation of a large number $n$ as the sum of $s$ quadratfrei numbers or, more generally, of $s$ numbers not divisible by $N$th powers greater than 1. $N \geqslant 2$ being fixed once and for all, we denote such a number generally by $M$ and call it a *power-free number* or *number of class* $N$. Let $\nu_s(n)$ be the number of representations of $n$ as the sum of $s$ power-free numbers, repetitions being allowed and order being regarded as relevant. It was shown in III, by purely arithmetical arguments, that

$$\nu_s(n) = \frac{n^{s-1}}{(s-1)!}\frac{1}{\zeta^s(N)}S(n)+O(n^{s-2+\frac{2}{N+1}+\epsilon}) \tag{1.1}$$

as $n \to \infty$, for every $\epsilon > 0$, where

$$S(n) = \prod_{p^N \nmid n}\left(1+\frac{(-1)^{s+1}}{(p^N-1)^s}\right)\prod_{p^N \mid n}\left(1+\frac{(-1)^s}{(p^N-1)^{s-1}}\right) \tag{1.2}$$

is greater than an absolute positive constant. On the other hand, an application of the Hardy-Littlewood method yielded in IV the error-term index

$$s-2+\frac{1}{N}+\frac{N-1}{N(s-1)}+\epsilon, \tag{1.3}$$

which is an improvement on (1.1) when $s \geqslant N+3$. The object of the present paper is the improvement of this last index. By the use of Gelbcke's refinement of the Winogradoff method, recently developed by him in connexion with Waring's problem,† we are able to avoid altogether the use of complex-function theory and to replace (1.3) when $s > \sqrt{N}+1$ by the sharper index

$$s-2+\frac{1}{N}+\frac{N-1}{Ns}+\epsilon. \tag{1.4}$$

This is an improvement on (1.1) when $s \geqslant N+2$.

* *Math. Zeitschrift*, 30 (1929), 433–48; *Journal für Math.* 164 (1931), 131–40; *Math. Zeitschrift*, 34 (1932), 637–44; *Annals of Mathematics*, 32 (1931), 261–70, referred to respectively as I, II, III, IV.

† M. Gelbcke, *Math. Annalen*, 105 (1931), 637–52. See also Landau, *Math. Zeitschrift*, 31 (1930), 319–38.

We shall assume a knowledge of §§ 2, 4, 5 of I; in particular we need the inequalities

$$\left|\frac{E_N(k)}{\zeta(N)}\prod_{p|k}\frac{-1}{p^N-1}\right| < \frac{\zeta(N)}{k}, \tag{1.5}$$

$$\left|\sum_{M\leqslant\nu} e^{2\pi i Ml/k} - (\nu+1)\frac{E_N(k)}{\zeta(N)}\prod_{p|k}\frac{-1}{p^N-1}\right| < A_1 k^{\epsilon}(\nu^{1/N}+k) \tag{1.6}$$

for $0 \leqslant l \leqslant k$, $(l,k) = 1$, where $E_N(k) = 1$ or $0$ according as $k$ is or is not a number of class $N+1$; $\epsilon$ denotes throughout an arbitrary positive number and assertions in which it occurs are to be understood as holding for every $\epsilon > 0$; $A_1, \ldots, A_{18}$ are positive constants depending only on $N$, $s$, $\epsilon$.

## 2. The Farey dissection; major and minor intervals

We divide up the interval $0 \leqslant t \leqslant 1$ by means of the Farey dissection of order $[n^K]$, where $\frac{1}{2} < K < 1$. That is to say, we take the Farey series of order $[n^K]$ in $(0,1)$ and separate its consecutive terms $\frac{l}{k}, \frac{l'}{k'}$ by the 'mediants' $\frac{l+l'}{k+k'}$. These mediants divide $(0,1)$ into sub-intervals $B_{l,k}$, each containing one term $\frac{l}{k}$ $(0 \leqslant l \leqslant k; (l,k) = 1)$ of the series. We agree to regard the two end-points of $(0,1)$ as identical, so that $k = 1$ gives rise to a single sub-interval $B_{0,1}$ whose two halves abut on 0 and 1 respectively. We can express this by calling $(0,1)$ a 'circular interval'. Writing

$$t = \frac{l}{k} + \vartheta, \tag{2.1}$$

we see that the interval $B_{l,k}$ is given by $-\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$, where by (2.1) of I

$$\frac{1}{2kn^K} \leqslant \vartheta_1, \vartheta_2 \leqslant \frac{1}{kn^K}. \tag{2.2}$$

The intervals $B_{l,k}$ may be divided into two classes, according to the values of the corresponding integers $k$. Those intervals $\mathfrak{M}$ for which

$$k \leqslant n^H,$$

where $H = 1-K$, will be called major intervals; the remainder $\mathfrak{m}$, for which $n^H < k \leqslant n^K$, minor intervals. Each major interval $\mathfrak{M}$ we divide up into a core $\mathfrak{M}_1$, in which

$$|\vartheta| \leqslant \frac{1}{2k^{\theta}n^{1-\theta H}}, \tag{2.3}$$

where $\theta = \dfrac{s-2}{s-1}$, and a fringe $\mathfrak{M}_2$, namely the pair of sub-intervals of $\mathfrak{M}$ in which

$$\frac{1}{2k^{\theta}n^{1-\theta H}} \leqslant |\vartheta| \leqslant \frac{1}{kn^{1-H}}. \tag{2.4}$$

In $\mathfrak{n}$ minor interval $\vartheta_1, \vartheta_2 < \dfrac{1}{n}$, so that $n|\vartheta| < 1$ throughout the interval.

## 3. The main argument

Let $M$ denote any number of class $N$ and let $\nu_s(n) = \nu(N, s; n)$ be the number of solutions of the Diophantine equation

$$M_1 + M_2 + \ldots + M_s = n \qquad (N \geqslant 2, s \geqslant 3),$$

where repetition of $M$'s is allowed and order is regarded as relevant. Then $\nu_s(n)$ is evidently the coefficient of $x^n$ in $f^s(x)$, where

$$f(x) = \sum_{M \leqslant n-1} x^M.$$

Hence, by an elementary calculation,

$$\nu_s(n) = \int_0^1 f^s(e^{2\pi i t}) e^{-2\pi i n t}\, dt. \tag{3.1}$$

To estimate this integral, we split up the range of integration by means of the Farey dissection already introduced and show that, when $n$ is large, $f(e^{2\pi i t})$ is approximately represented in the cores $\mathfrak{M}_1$ of the respective major intervals $\mathfrak{M}$ by certain polynomials $\psi_{l,k}(e^{2\pi i t})$ in $e^{2\pi i t}$ of very simple type, while in the fringes $\mathfrak{M}_2$ and in each of the minor arcs $\mathfrak{m}$, $f(x)$ is relatively small, so that it may be replaced by zero to the same order of approximation.

Since $$|f(e^{2\pi i t})| < n$$

and $$\int_0^1 |f(e^{2\pi i t})|^2\, dt = \sum_{M \leqslant n-1} 1 < n,$$

we have $$\int_0^1 |f(e^{2\pi i t})|^{s-1}\, dt < n^{s-2}.$$

We proceed to the estimation of the integral (3.1).

## 4. Major intervals

4.1. *Cores*

Let $0 \leqslant l \leqslant k; (l,k) = 1; \rho = e^{2\pi i l/k}; e^{2\pi i t} = e^{2\pi i (l/k + \vartheta)} = \rho e^{2\pi i \vartheta} = \rho X$ and let

$$\psi_{l,k}(e^{2\pi i t}) = (1 + X + \ldots + X^{n-1}) \frac{E_N(k)}{\zeta(N)} \prod_{p|k} \frac{-1}{p^N - 1}.$$

Then

$$\psi_{l,k}(e^{2\pi il})$$

$$= (1-X)\sum_{\nu=0}^{n-1}(\nu+1)\frac{E_N(k)}{\zeta(N)}\prod_{p|k}\frac{-1}{p^N-1}X^\nu + n\frac{E_N(k)}{\zeta(N)}\prod_{p|k}\frac{-1}{p^N-1}X^n.$$

But
$$f(e^{2\pi il}) = \sum_{M\leqslant n-1} e^{2\pi iMl} = \sum_{M\leqslant n-1}\rho^M X^M$$

$$= (1-X)\sum_{\nu=0}^{n-1}\tau(\nu)X^\nu + \tau(n-1)X^n,$$

where $\tau(\nu) = \sum_{M\leqslant\nu}\rho^M$. Thus, by (2.8) of I,

$$|f(e^{2\pi il})-\psi_{l,k}(e^{2\pi il})| < |1-X|\sum_{\nu=0}^{n-1}A_2\,k^\epsilon(\nu^{1/N}+k)+A_2\,k^\epsilon(n^{1/N}+k).$$

Further,
$$|1-X| = |1-e^{2\pi i\vartheta}| < 2\pi|\vartheta|.$$

Thus in the circular interval $(0, 1)$

$$|f(e^{2\pi il})-\psi_{l,k}(e^{2\pi il})| < A_3\,k^\epsilon(1+n|\vartheta|)(n^{1/N}+k). \tag{4.11}$$

Now suppose that $k \leqslant n^H$ and $t$ lies in the core of the major arc corresponding to $l, k$. Then by (2.3)

$$1+n|\vartheta| \leqslant 1+\frac{1}{2}\frac{n^{H\theta}}{k^\theta} < 2\frac{n^{H\theta}}{k^\theta},$$

and
$$|f(e^{2\pi il})-\psi_{l,k}(e^{2\pi il})| < A_4\,n^\epsilon(n^{H\theta+1/N}+n^{H-H\theta+H\theta})$$

$$= A_4\,n^\epsilon(n^{H\theta+1/N}+n^H). \tag{4.12}$$

LEMMA 4.13.

$$\sum_{B_{l,k}}\int_{B_{l,k}}|\psi_{l,k}(e^{2\pi il})|^{s-1}\,dt < A_5\,n^{s-2+\epsilon}.$$

PROOF.

$$\psi_{l,k}(e^{2\pi il}) = \frac{1-e^{2\pi in\vartheta}}{1-e^{2\pi i\vartheta}}\frac{E_N(k)}{\zeta(N)}\prod_{p|k}\frac{-1}{p^N-1}.$$

Thus by (1.5)

$$|\psi_{l,k}(e^{2\pi il})| < \frac{\zeta(N)}{k}\left|\frac{\sin\pi n\vartheta}{\sin\pi\vartheta}\right|,$$

$$\int_0^1|\psi_{l,k}(e^{2\pi il})|^{s-1}\,dt < \frac{\zeta^{s-1}(N)}{k^{s-1}}\,n^{s-3}\int_0^1\frac{\sin^2\pi n\vartheta}{\sin^2\pi\vartheta}\,d\vartheta$$

$$= \frac{\zeta^{s-1}(N)}{k^{s-1}}\,n^{s-2};$$

$$\sum_{B_{l,k}} \int_{B_{l,k}} |\psi_{l,k}(e^{2\pi it})|^{s-1}\,dt < \sum_{B_{l,k}} \int_0^1 |\psi_{l,k}(e^{2\pi it})|^{s-1}\,dt$$

$$< \zeta^{s-1}(N) n^{s-2} \sum_{k=1}^{n^K} \frac{\phi(k)}{k^{s-1}} < A_5\, n^{s-2+\epsilon},$$

which proves the lemma.

Now

$$\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s(e^{2\pi it}) - \psi_{l,k}^s(e^{2\pi it})|\,dt$$

$$\leqslant s \max_{\mathfrak{M}_1} |f(e^{2\pi it}) - \psi_{l,k}(e^{2\pi it})| \times$$

$$\times \Big( \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f(e^{2\pi it})|^{s-1}\,dt + \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |\psi_{l,k}(e^{2\pi it})|^{s-1}\,dt \Big).$$

The first of these sums is less than

$$\int_0^1 |f(e^{2\pi it})|^{s-1}\,dt < n^{s-2}.$$

The second is less than $A_5\, n^{s-2+\epsilon}$, by Lemma 4.13. Thus from (4.12) we have

$$\sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} |f^s - \psi_{l,k}^s|\,dt < A_6\, n^{s-2+\epsilon}(n^{H\theta+1/N} + n^H). \tag{4.14}$$

4.2. *Fringes.**

Next suppose that $t$ lies in the fringe of the major interval corresponding to $l, k$. Thus

$$1 \leqslant k \leqslant n^K, \qquad 0 \leqslant l \leqslant k, \qquad (l,k) = 1, \qquad t = \frac{l}{k} + \vartheta$$

and

$$\frac{1}{2k^\theta n^{1-\theta H}} < |\vartheta| < \frac{1}{kn^{1-H}},$$

so that

$$\frac{2}{k|\vartheta|} > 2n^{1-H} > 1.$$

It follows by a well-known result in the theory of Farey series† that we can find $l_1$, $k_1$, $\vartheta_1$ such that

$$0 \leqslant l_1 \leqslant k_1 \leqslant \frac{2}{k|\vartheta|}, \qquad (l_1, k_1) = 1, \qquad \frac{l_1}{k_1} + \vartheta_1 = \frac{l}{k} + \vartheta$$

* It is here that the Winogradoff argument is applied; it amounts to the introduction of an auxiliary Farey series to deal with the fringes. Gelbcke's refinement consists in making the division into cores and fringes depend on $k$ as well as on $n$.

† See Landau, *Vorlesungen über Zahlentheorie*, i, Leipzig, 1927, Satz 159.

and $$|\vartheta_1| < \frac{k|\vartheta|}{2k_1} \leqslant \frac{1}{k_1^2}.$$

We proceed to consider the possible value of $k_1$. In the first place, $n^H \leqslant k_1 \leqslant n^{1-H}$ is impossible, for this makes

$$|\vartheta_1| < \frac{k|\vartheta|}{2k_1} < \frac{1}{2k_1 n^{1-H}}$$

and $t$ would then lie in a minor arc, by (2.2). Secondly, $k_1 < n^H$ is impossible, for it gives

$$\begin{aligned} |lk_1 - l_1 k| &= kk_1\left|\frac{l}{k} - \frac{l_1}{k_1}\right| = kk_1|\vartheta - \vartheta_1| \\ &\leqslant kk_1|\vartheta| + kk_1|\vartheta_1| < k_1 k|\vartheta| + k^2|\vartheta| = (k+k_1)k|\vartheta| \\ &\leqslant (k+k_1)n^{-1+H}, \qquad \text{by (2.2)}, \\ &\leqslant 2n^{2H-1} < 1 \end{aligned}$$

for $n \geqslant n_0(H)$. This gives, for $n \geqslant n_0$,

$$lk_1 = l_1 k; \qquad k = k_1,\ l = l_1,\ \vartheta = \vartheta_1; \qquad |\vartheta_1| = \frac{k|\vartheta|}{k_1} > \frac{k|\vartheta|}{2k_1},$$

and we have a contradiction. It follows that for $n \geqslant n_0$

$$k_1 > n^{1-H}. \tag{4.21}$$

Also, by (2.4), $$k_1 \leqslant \frac{2}{k|\vartheta|} < \frac{4n^{1-\theta H}}{k^{1-\theta}} \leqslant 4n.$$

Now $$f(e^{2\pi i t}) = \sum_{M \leqslant n-1} e^{2\pi i M t} = \sum_{M \leqslant n-1} e^{2\pi i M l_1/k_1} X_1^M$$

$$= (1-X_1)\sum_{\nu=\sigma}^{n-1} \sigma(\nu) X_1^\nu + \sigma(n-1)X_1^n,$$

where $X_1 = e^{2\pi i \vartheta_1}$ and

$$\sigma(\nu) = \sum_{N \leqslant \nu} e^{2\pi i M l_1/k_1}.$$

Thus, by (1.5), (1.6),

$$|\sigma(\nu)| < A_7 n^\epsilon\left(\frac{\nu}{k_1} + \nu^{1/N} + k_1\right),$$

and, for $\nu \leqslant n-1$,

$$|\sigma(\nu)| < A_7 n^\epsilon\left(\frac{n}{k_1} + n^{1/N} + \frac{2}{k|\vartheta|}\right) < A_7 n^\epsilon\left(n^H + n^{1/N} + \frac{2}{k|\vartheta|}\right).$$

Since $$|1-X_1| = |1-e^{2\pi i\vartheta_1}| < \pi|\vartheta_1| < \frac{\pi}{k_1^2} < \frac{\pi}{n^{2-2H}},$$

it follows that in a pair of fringe intervals $\mathfrak{M}_2$

$$|f(e^{2\pi i t})| < A_7 n^\epsilon\left(\frac{\pi}{n^{1-2H}} + 1\right)\left(n^H + n^{1/N} + \frac{2}{k|\vartheta|}\right). \tag{4.22}$$

We can now estimate $\sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f(e^{2\pi i t})|^s \, dt$.

By (4.22) it is less than

$$A_8 n^\epsilon (n^{sH} + n^{s/N}) \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} dt + A_9 n^\epsilon \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} \frac{dt}{k^s |\vartheta|^s}.$$

The first term is less than $A_{10} n^\epsilon (n^{sH} + n^{s/N})$; the second is less than

$$\begin{aligned} & 2A_9 n^\epsilon \sum_{\mathfrak{M}_2} \frac{1}{k^s} \int_{\frac{1}{2}k^{-\theta} n^{\theta H - 1}}^{\infty} \frac{d\vartheta}{\vartheta^s} \\ &= A_{10} n^\epsilon \sum_{\mathfrak{M}_2} \frac{1}{k^s} k^{(s-1)\theta} n^{(s-1)(1-\theta H)} \\ &= A_{10} n^{(s-1)(1-\theta H)+\epsilon} \sum_{k \leqslant n^H} \frac{\phi(k)}{k^{s-s\theta} h^\theta} \\ &< A_{11} n^{(s-1)(1-\theta H)+\epsilon}, \end{aligned}$$

since $s - s\theta + \theta = 2$ and $\phi(k) \leqslant k$. Thus

$$\sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} |f(e^{2\pi i t})|^s \, dt < A_{12} n^\epsilon (n^{sH} + n^{s/N} + n^{s-1-(s-2)H}). \tag{4.23}$$

## 5. Minor intervals

By (1.5), (1.6),

$$\left| \sum_{M \leqslant \nu} e^{2\pi i M l/k} \right| < A_1 k^\epsilon \left( \frac{\nu}{k} + \nu^{1/N} + k \right).$$

Thus for $\nu < n, \quad n^H < k \leqslant n^K$

$$|\tau(\nu)| < A_1 n^\epsilon (n^{1-H} + n^{1/N} + n^K) < A_{13} n^{K+\epsilon}.$$

It follows that in a minor interval

$$\begin{aligned} |f(e^{2\pi i \vartheta})| &\leqslant |1 - e^{2\pi i \vartheta}| \left| \sum_{\nu=0}^{n-1} \tau(\nu) e^{2\pi i \nu \vartheta} \right| + |\tau(n-1) e^{2\pi i n \vartheta}| \\ &< |1 - e^{2\pi i \vartheta}| \sum_{\nu=0}^{n-1} A_{13} n^{K+\epsilon} + A_{13} n^{K+\epsilon} \\ &< A_{14} n |\vartheta| n^{K+\epsilon} + A_{13} n^{K+\epsilon} \\ &< A_{15} n^{K+\epsilon}. \end{aligned}$$

Whence

$$\begin{aligned} \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f(e^{2\pi i t})|^s \, dt &< A_{16} n^{(s-2)K+\epsilon} \sum_{\mathfrak{m}} \int_{\mathfrak{m}} |f(e^{2\pi i t})|^2 \, dt \\ &< A_{16} n^{(s-2)K+1+\epsilon} \\ &= A_{16} n^{s-1-(s-2)H+\epsilon}. \end{aligned} \tag{5.1}$$

## 6. Proof of (1.4)

Starting from (3.1), we have

$$\nu_s(n) = \int_0^1 f^s(e^{2\pi it})e^{-2\pi int}\,dt$$

$$= \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_1} \psi_{l,k}^s(e^{2\pi it})e^{-2\pi int}\,dt + \lambda_1(n), \tag{6.1}$$

where

$$\lambda_1(n) = \sum_{\mathfrak{M}_1} \int_{\mathfrak{M}_2} (f^s - \psi_{l,k})e^{-2\pi int}\,dt + \sum_{\mathfrak{M}_2} \int_{\mathfrak{M}_2} f_1^s e^{-2\pi int}\,dt + \sum_{\mathfrak{m}} \int_{\mathfrak{m}} f^s e^{-2\pi int}\,dt$$

$$= \lambda_1^{(1)}(n) + \lambda_1^{(2)}(n) + \lambda_1^{(3)}(n),$$

say. By (4.14)

$$|\lambda_1^{(1)}(n)| < A_6\, n^{s-2+\epsilon}(n^{H\theta+1/N} + n^H);$$

by (4.23)

$$|\lambda_1^{(2)}(n)| < A_{12}\, n^{\epsilon}(n^{sH} + n^{s/N} + n^{s-1-(s-2)H});$$

and by (5.1)

$$|\lambda_1^{(3)}(n)| < A_{16}\, n^{\epsilon} n^{s-1-(s-2)H}.$$

We next replace each $\int_{\mathfrak{M}_1}$ in (6.1) by the integral over the whole of $(0, 1)$. The resulting error

$$\lambda_2(n) = \sum_{\mathfrak{M}_1} \int_{(0,1)-\mathfrak{M}_1} \psi_{l,k}^s(e^{2\pi it})e^{-2\pi int}\,dt$$

satisfies the inequality

$$|\lambda_2(n)| < \zeta^s(N) \sum_{\mathfrak{M}_1} \frac{1}{k^s} \int_{(-\frac{1}{k}, \frac{1}{k})-\mathfrak{M}_1} \left| \frac{\sin^s \pi n\vartheta}{\sin^s \pi\vartheta} \right| d\vartheta$$

$$< 2\zeta^s(N) \sum_{k \leqslant n^H} \frac{\phi(k)}{k^s} \int_{\frac{1}{2}k^{-c}n^{\theta H-1}}^{\infty} \frac{d\vartheta}{\vartheta^s}$$

$$< c_1\, n^{(1-\theta H)(s-1)} \sum_{k \leqslant n^H} \frac{1}{k^{(1-\theta)(s-1)}},$$

where $c_1 > 0$ depends only on $N$, $s$,

$$= c_1\, n^{(s-1)-(s-2)H} \sum_{k \leqslant n^H} \frac{1}{k}$$

$$< A_{17}\, n^{(s-1)-(s-2)H+\epsilon}.$$

Our integral has now been replaced by

$$\sum_{\mathfrak{M}_1} \int_0^1 \psi_{l,k}^s(e^{2\pi it})e^{-2\pi int}\,dt, \tag{6.2}$$

and in this sum $\int_0^1$ is simply the coefficient of $x^n$ in the expansion of $\psi_{l,k}^s(x)$. But this is the same as the coefficient of $x^n$ in the expansion of $\psi_\rho^s(x)$ of I, since $\psi_{l,k}(x)$ and $\psi_\rho(x)$ agree in their first $n$ terms. It follows that (6.2) is equal to (4.3) of I and so differs from

$$\frac{n^{s-1}}{(s-1)!}\frac{1}{\zeta^s(N)}S(n)$$

by $\lambda_3(n)$, where, by an easy calculation,

$$|\lambda_3(n)| < A_{18}\, n^{s-1-(s-2)H+\epsilon}.$$

It remains to choose $H$ so that $\lambda_1(n)$, $\lambda_2(n)$, $\lambda_3(n)$ are all

$$O\Big(n^{s-2+\frac{1}{N}+\frac{N-1}{Ns}+\epsilon}\Big).$$

This we do by taking $H = \dfrac{N-1}{Ns}\dfrac{s-1}{s-2}$. Then, since $s > \sqrt{N}+1$, $H < \frac{1}{2}$ and

$$H\theta+\frac{1}{N} = \frac{1}{N}+\frac{N-1}{Ns}, \quad H = \frac{1}{N}+\frac{N-1}{Ns}+\frac{N-(s-1)^2}{Ns(s-2)} < \frac{1}{N}+\frac{N-1}{Ns},$$

$$1-(s-2)H = \frac{1}{N}+\frac{N-1}{Ns},$$

$$sH-(s-2) = 1-(s-2)H-(s-1)(1-2H)$$

$$<1-(s-2)H = \frac{1}{N}+\frac{N-1}{Ns},$$

$$\frac{s}{N}-(s-2) = \frac{1}{N}+\frac{N-1}{Ns}-\frac{(N-1)(s-1)^2-s}{Ns} < \frac{1}{N}+\frac{N-1}{Ns}.$$

The proof is therefore complete.